MATH 790, FALL 2011, HOMEWORK 4
DUE FRIDAY 30 SEPTEMBER

Let R be a commutative noetherian ring. Then $\mathrm{Spec}(R)$ is the set of all prime ideals of R. For each subset $S \subseteq R$, set $V(S) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid S \subseteq \mathfrak{p}\}$. For each element $f \in R$, set $D(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) \mid f \notin \mathfrak{p}\}$.

Exercise 1. Let $S \subseteq R$ be a subset. Let $\{I_j\}_{j\in J}$ be a set of ideals of R.
(a) Prove that $V(0) = \mathrm{Spec}(R)$ and $V(1) = \emptyset$.
(b) Prove that $V(S) = V((S)R)$.
(c) Prove that if J is finite then $V(\cap_{j\in J} I_j) = \cup_{j\in J} V(I_j)$.
(d) Prove that $V(\cup_{j\in J} I_j) = V(\sum_{j\in J} I_j) = \cap_{j\in J} V(I_j)$.

This shows that the set $\{V(I) \mid I \subseteq R \text{ is an ideal}\} = \{V(S) \mid S \subseteq R \text{ is a subset}\}$ satisfies the axioms to be the set of closed sets in a topology on $\mathrm{Spec}(R)$. This topology is the *Zariski topology* named after the famous algebraic geometer Oscar Zariski.

Exercise 2. Let $S \subseteq R$ be a subset. Let $\{I_j\}_{j\in J}$ be a set of ideals of R.
(a) Prove that $V(I) = V(\mathrm{rad}(I))$ where

$$\mathrm{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n \geqslant 1\} = \bigcap_{\mathfrak{p}\in V(I)} \mathfrak{p}$$

is the radical of I.
(b) Prove that for each $f \in R$, the set $D(f)$ is an open set in the Zariski topology.
(c) Prove that for every open set $U \subseteq \mathrm{Spec}(R)$, there is a subset $T \subseteq R$ such that $U = \cup_{t\in T} D(t)$. In other words, the set $\{D(f) \mid f \in R\}$ forms a basis for the Zariski topology on $\mathrm{Spec}(R)$.

Exercise 3. [Extra Credit]
(a) Prove that if R is noetherian, then every open subset $U \subseteq \mathrm{Spec}(R)$ is compact, i.e., given any open cover $U = \cup_{x\in X} U_x$ there is a finite subcover $U = \cup_{i=1}^{g} U_{x_i}$. In particular, $\mathrm{Spec}(R)$ is compact in this case.
[A topological space X is *noetherian* if every open subset is compact. This is equivalent to satisfying ACC on open sets, which is equivalent to satisfying DCC on open sets.]
(b) Find an example of a non-noetherian ring R such that $\mathrm{Spec}(R)$ is finite. In particular, this shows that the converse of part (a) fails in general.
(c) Let $f\colon R \to R'$ be a ring homomorphism. Define $\mathrm{Spec}(f)\colon \mathrm{Spec}(R') \to \mathrm{Spec}(R)$ by the formula $\mathrm{Spec}(f)(\mathfrak{p}') := f^{-1}(\mathfrak{p}')$. Prove that $\mathrm{Spec}(f)$ is well-defined and continuous.
[Note that $\mathrm{Spec}(f)$ is usually denoted f^*.]
(d) Prove that $\mathrm{Spec}(-)$ is a contravariant functor from the category of commutative rings with identity (with ring homomorphisms) to the category of topological spaces (with continuous maps).