MATH 790, FALL 2011, HOMEWORK 6
DUE FRIDAY 21 OCTOBER

Exercise 1. Let k be a field, and set $R = k[X, Y]$.
(a) For $m, n \geqslant 1$, we have
$$\mathrm{Supp}_R(R/(X^m, Y^n)R) = \{(X, Y)R\} = \mathrm{Supp}_R(R/((X, Y)R)^n).$$
(b) For $M = R/(X^2, XY)R$, we have
$$\mathrm{Supp}_R(R/(X^2, XY)R) = \mathrm{Supp}_R(R/(X)R) = \{P \in \mathrm{Spec}(R) \mid X \in P\}.$$
[Hint: Remark V.2.6 from the homological algebra notes.]

Exercise 2. Let R be a commutative ring, and let $r \in R$ be an R-regular element, that is, a non-unit that is not a zero-divisor on R. Prove that
$$\mathrm{Ass}_R(R/r^nR) = \mathrm{Ass}_R(R/rR)$$
for all $n \geqslant 1$. [Hint: Verify that the following sequence is exact:
$$0 \to R/rR \xrightarrow{r^{n-1}} R/r^nR \to R/r^{n-1}R \to 0$$
and use induction on n. Feel free to use Proposition V.2.13(b) from the homological algebra notes.]

Exercise 3. Let R be a unique factorization domain, and let $0 \neq r \in R$ be a non-unit. Write $r = p_1 \cdots p_n$ with each p_i prime. Prove that
$$\mathrm{Ass}_R(R/rR) = \{p_1R, \ldots, p_nR\}$$
as follows. Check that the following is a prime filtration of R/rR
$$(0) = (p_1 \cdots p_nR)/rR \subsetneq (p_1 \cdots p_{n-1}R)/rR \subsetneq \cdots \subsetneq p_1R/rR \subsetneq R/rR$$
by showing that
$$\frac{(p_1 \cdots p_iR)/rR}{(p_1 \cdots p_{i+1}R)/rR} \cong \frac{p_1 \cdots p_iR}{p_1 \cdots p_{i+1}R} \cong \frac{R}{p_{i+1}R}.$$
Explain why this gives the containment $\mathrm{Ass}_R(R/rR) \subseteq \{p_1R, \ldots, p_nR\}$. Next, set $p_i' = \prod_{j \neq i} p_i$ for each i, and define a function $R/p_iR \to R/rR$ given by $\overline{x} \mapsto \overline{p_i'x}$. Show that this is a well-defined monomorphism, and show that this implies that $\mathrm{Ass}_R(R/rR) \supseteq \{p_1R, \ldots, p_nR\}$.