MATH 790, FALL 2011, HOMEWORK 7
DUE FRIDAY 21 OCTOBER

Exercise 1. Let k be a field, and set $R = k[X, Y]$.

(a) Prove that for all $m, n \geqslant 1$, we have

$$\mathrm{Ass}_R(R/(X^m, Y^n)R) = \{(X, Y)R\} = \mathrm{Ass}_R(R/((X, Y)R)^n).$$

[Hint: Feel free to use V.2.11(c) from the homological algebra notes.]

(b) For $M = R/(X^2, XY)R$, we have

$$\mathrm{Ass}_R(R/(X^2, XY)R) = \{(X)R, (X, Y)R\}.$$

[Hint: For one containment, show that there is a filtration of $R/(X^2, XY)R$ where the quotients M_i/M_{i-1} are isomorphic to $R/(X)$ and $R/(X, Y)$.]

(c) Prove or give a counterexample to the following:

$$\mathrm{ZD}_R(\mathrm{Hom}_R(R/(X^2, XY)R, R/(X^m, Y^n)R)) = (X, Y)R.$$

(d) For $M = R/(X^2, XY)$, prove that for each integer $n \geqslant 1$, there is a prime filtration of M over R such that the prime ideal (X, Y) occurs exactly n times in the filtration. (In particular, this shows that the number of "links" in a prime filtration is dependent on the choice of prime filtration. Contrast this with the Jordan-Hölder Theorem.)