MATH 790, FALL 2011, HOMEWORK 8
DUE FRIDAY 28 OCTOBER

Fact 1. Let R be a commutative local ring, and let P be a finitely generated projective R-module. Then P is free over R, say $P \cong R^n$. (We will prove this in class. Note that R satisfies the invariant basis property, so n is the only integer m such that $P \cong R^m$.) The number n is the *rank* of R, denoted $\operatorname{rank}_R(P)$.

Exercise 2. Let R be a commutative ring, and let Q be a finitely generated projective R-module.

(a) Prove that for each $\mathfrak{p} \in \operatorname{Spec}(R)$, the localization $Q_\mathfrak{p}$ is a finitely generated projective $R_\mathfrak{p}$-module, so we have $Q_\mathfrak{p} \cong R_\mathfrak{p}^{e_\mathfrak{p}}$ where $e_\mathfrak{p} = \operatorname{rank}_{R_\mathfrak{p}}(Q_\mathfrak{p})$.

(b) Prove that for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$, if $\mathfrak{q} \subseteq \mathfrak{p}$, then $\operatorname{rank}_{R_\mathfrak{q}}(Q_\mathfrak{q}) = \operatorname{rank}_{R_\mathfrak{p}}(Q_\mathfrak{p})$, that is, $e_\mathfrak{q} = e_\mathfrak{p}$.

Definition 3. Let R be a commutative ring, and let Q be a finitely generated projective R-module. The *rank function* for Q is the function $r_Q \colon \operatorname{Spec}(R) \to \mathbb{N}$ defined as $r_Q(\mathfrak{p}) := \operatorname{rank}_{R_\mathfrak{p}}(Q_\mathfrak{p})$.

Exercise 4. Let R be a commutative noetherian ring, and let Q be a finitely generated projective R-module. Let C be a semidualizing R-module. Prove that the following conditions are equivalent:

(i) $Q \otimes_R C$ is a semidualizing R-module;
(ii) $r_Q(\mathfrak{p}) = 1$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$; and
(iii) $r_Q(\mathfrak{m}) = 1$ for each maximal ideal $\mathfrak{m} \subset R$.

Exercise 5. (bonus) Let R be a commutative noetherian ring, and let Q be a finitely generated projective R-module. For each integer $n \geqslant 0$ set

$$X_n = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid r_Q(\mathfrak{p}) = n\}.$$

Let m be a non-negative integer.

(a) Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $r_Q(\mathfrak{p}) = m$. Prove that there is an element $s \in R \smallsetminus \mathfrak{p}$ such that $Q_s \cong R_s^m$.

(b) Prove that X_m is an open subset of $\operatorname{Spec}(R)$ in the Zariski topology.

(c) Observe that the compliment $\operatorname{Spec}(R) \smallsetminus X_m$ is the union $\bigcup_{n \neq m} X_n$ of open sets, so it is open. Conclude that X_m is a closed subset of $\operatorname{Spec}(R)$ in the Zariski topology.