MATH 790, FALL 2011, HOMEWORK 9–10
DUE FRIDAY 18 NOVEMBER

Exercise 1. Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring, and let C be a semidualizing R-module. Prove that the following conditions are equivalent.

(i) $C \cong R$;
(ii) $\mathcal{A}_C(R)$ contains every R-module;
(iii) $k \in \mathcal{A}_C(R)$;
(iv) $\mathcal{B}_C(R)$ contains every R-module;
(v) $k \in \mathcal{B}_C(R)$.

Definition 2. Let R be a ring, and let I be an injective R-module. Then I is *faithfully injective* if, for every sequence S of R-module homomorphisms such that $\mathrm{Hom}_R(S, I)$ is exact, the sequence S must be exact. (Compare this to the definition of "faithfully flat".)

Fact 3. Let R be a ring, and let I be an injective R-module. The following conditions are equivalent.

(i) I is faithfully injective;
(ii) for each R-module $M \neq 0$, we have $\mathrm{Hom}_R(M, I) \neq 0$;
(iii) for each maximal ideal $\mathfrak{m} \subset R$, we have $\mathrm{Hom}_R(R/\mathfrak{m}, I) \neq 0$.

Exercise 4. (Bonus) Prove Fact 3.

Exercise 5. Let R be a commutative ring.

(a) Prove that if F is a flat R-module and I is an injective R-module, then $\mathrm{Hom}_R(F, I)$ is an injective R-module.
(b) Prove that if M is an R-module and I is a faithfully injective R-module such that $\mathrm{Hom}_R(F, I)$ is an injective R-module, then M is a flat R-module.

Exercise 6. Let R be a commutative noetherian ring. Let M be a finitely generated R-module, and let $\{N_i\}_{i \in I}$ be a set of R-modules.

(a) Prove that there is an isomorphism $M \otimes_R (\prod_{i \in I} N_i) \xrightarrow{\cong} \prod_{i \in I}(M \otimes_R N_i)$. (Hint: Prove it first for R, then for R^n, then in general using the right exactness of tensor product.)
(b) Prove that there is an isomorphism $\oplus_{i \in I} \mathrm{Hom}_R(M, N_i) \xrightarrow{\cong} \mathrm{Hom}_R(M, \oplus_{i \in I} N_i)$. (Hint: Modify the hint from part (a).)