MATH 720, Algebra I
Exercises 1
Due Wed 10 Sep

Exercise 1. Let $n \in \mathbb{N} = \{1, 2, \ldots\}$, and let $f\colon \mathbb{Z}^n \to \mathbb{Q}$ be a homomorphism of additive abelian groups.

(a) Prove that f is not an epimorphism.
(b) Prove that if $n \geq 2$, then f is not a monomorphism.
(c) Prove that there is an isomorphism of additive abelian groups $\mathbb{Z}^{(\mathbb{N})}/K \cong \mathbb{Q}$ for some subgroup $K \leqslant \mathbb{Z}^{(\mathbb{N})}$. (Hint: Since $\mathbb{Q}$ is countable, we can write $\mathbb{Q} = \{a_1, a_2, \ldots\}$. Consider the function $g\colon \mathbb{Z}^{(\mathbb{N})} \to \mathbb{Q}$ given by $g(r_1, r_2, \ldots) := \sum_i r_i a_i$.)

Exercise 2. Consider the fields $\mathbb{R}$ and $\mathbb{C}$.

(a) Prove that there is an isomorphism of additive abelian groups $f\colon \mathbb{R} \times \mathbb{R} \xrightarrow{\cong} \mathbb{C}$.
(b) Prove or disprove: f is an isomorphism of rings.

Exercise 3. Consider the ring $M_2(\mathbb{R})$ and the following subset.

$$S = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in M_2(\mathbb{R}) \,\middle|\, a, b \in \mathbb{R} \right\}$$

(a) Prove that S is a commutative subring of $M_r(\mathbb{R})$ with multiplicative identity $1_S = 1_{M_2(\mathbb{R})}$.
(b) Prove that there is an isomorphism $f\colon \mathbb{C} \xrightarrow{\cong} S$ such that $f(1) = 1_S$.
(c) Prove that S is a field.
(d) Prove or disprove: $M_2(\mathbb{R})$ is a field.