MATH 720, Algebra I
Exercises 10
Due Wed 19 Nov

Throughout this homework set, let $R$ be a ring with identity.

**Exercise 1.** Let $I$ be a two-sided ideal of $R$, and let $M$ be a unitary $R$-module. Let $IM$ be the submodule of $M$ generated by the set $\{im \in M \mid i \in I \text{ and } m \in M\}$.

(a) Prove that $IM = \{\sum_n^{\text{finite}} i_n m_n \mid i_n \in I \text{ and } m_n \in M\}$.
(b) Prove that $M/IM$ has a well-defined unitary $R/I$-module structure given by $(r+I)(m+IM) := (rm) + IM$.

**Exercise 2.** A unitary $R$-module $M$ is *cyclic* if there is an element $m \in M$ such that $M = \langle m \rangle$. Prove that a unitary $R$-module $M$ is cyclic if and only if there is a left ideal $I \subseteq R$ such that $M \cong R/I$.

**Exercise 3.** A non-zero unitary $R$-module $M$ is *simple* if its only submodules are $0$ and $M$.

(a) Prove that if $M$ is a simple $R$-module, then $M$ is cyclic.
(b) Prove or disprove the converse to part (a).
(c) Let $f\colon M \to N$ be a homomorphism between simple $R$-modules. Prove that $f$ is either $0$ or an isomorphism.
(d) Assume that $R$ is commutative. Prove that $M$ is a simple $R$-module if and only if there is a maximal ideal $\mathfrak{m} \subset R$ such that $M \cong R/\mathfrak{m}$.