MATH 720, Algebra I
Exercises 11
Due Wed 26 Nov

Throughout this homework set, let R be a commutative ring with identity. Let L, M, and N be unitary R-modules.

Exercise 1. Let $f\colon M \to N$ be an R-module isomorphism. Prove that the map $f^*\colon \operatorname{Hom}_R(N,L) \to \operatorname{Hom}_R(M,L)$ is an isomorphism.

Exercise 2 (The Five Lemma)**.** Consider the following commutative diagram of R-module homomorphisms with exact rows

$$\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\
B_1 & \xrightarrow[g_1]{} & B_2 & \xrightarrow[g_2]{} & B_3 & \xrightarrow[g_3]{} & B_4 & \xrightarrow[g_4]{} & B_5
\end{array}$$

(a) Prove that if α_1 is an epimorphism and α_2, α_4 are monomorphisms, then α_3 is a monomorphism.
(b) Prove that if α_5 is a monomorphism and α_2, α_4 are epimorphisms, then α_3 is an epimorphism.

Exercise 3. Consider the following commutative diagram of R-module homomorphisms with exact rows

$$\begin{array}{ccccccccc}
 & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & 0 \\
 & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
0 & \longrightarrow & B_1 & \xrightarrow[g_1]{} & B_2 & \xrightarrow[g_2]{} & B_3 & &
\end{array}$$

(a) Prove that there is an exact sequence $\operatorname{Ker}(\alpha_1) \to \operatorname{Ker}(\alpha_2) \to \operatorname{Ker}(\alpha_3)$.
(b) Prove that there is an exact sequence $B_1/\operatorname{Im}(\alpha_1) \to B_2/\operatorname{Im}(\alpha_2) \to B_3/\operatorname{Im}(\alpha_3)$.
(c) (Bonus) Show how this gives another proof of Proposition 3.7.9 from the notes.