MATH 720, Algebra I
Exercises 12
Due Wed 03 Dec

Throughout this homework set, let R and S be commutative rings with identity, and let M be a unitary R-module.

Exercise 1. Prove that the following conditions are equivalent:

(i) M is noetherian as an R-module,
(ii) every submodule $N \subseteq M$ is noetherian as an R-module, and
(iii) for every submodule $N \subseteq M$, the quotient M/N is noetherian as an R-module.

Exercise 2. Prove that the following conditions are equivalent:

(i) R is a noetherian ring,
(ii) for every ideal $I \subseteq R$, the quotient R/I is a noetherian ring,
(iii) for every $n \in \mathbb{N}_+$ the polynomial ring $R[x_1, \ldots, x_n]$ is a noetherian ring, and
(iv) for some $n \in \mathbb{N}_+$ the polynomial ring $R[x_1, \ldots, x_n]$ is a noetherian ring.

Exercise 3 (Bonus)**.** Let $S \to R$ be a ring epimorphism. Prove that M is noetherian as an S-module if and only if M is noetherian as an R-module.

Exercise 4 (Bonus)**.** Consider an exact sequence of R-module homomorphisms

$$0 \to M' \xrightarrow{f'} M \xrightarrow{f} M'' \to 0.$$

Prove that M is noetherian if and only if M' and M'' are both noetherian.