MATH 720, Algebra I
Exercises 13
Due Fri 12 Dec

Throughout this homework set, let $R$ be a commutative ring with identity.

**Exercise 1.** Let $L$, $M$, and $N$ be unitary $R$-modules. Let $i\colon L \to L\oplus M$ be defined as $i(x) = (x, 0)$. Let $j\colon M \to L \oplus M$ be defined as $j(y) = (0, y)$. Define

$$\Phi\colon \operatorname{Hom}_R(L \oplus M, N) \to \operatorname{Hom}_R(L, N) \oplus \operatorname{Hom}_R(M, N)$$

by the formula $\Phi(f) = (f \circ i, f \circ j)$. Define

$$\Psi\colon \operatorname{Hom}_R(L, N) \oplus \operatorname{Hom}_R(M, N) \to \operatorname{Hom}_R(L \oplus M, N)$$

by the formula $\Psi(g, h) = g \boxplus h$ where $(g \boxplus h)(x, y) = g(x) + h(y)$.

(a) Prove that $\Phi$ is an $R$-module isomorphism with inverse $\Psi$.

(b) (Bonus) Prove that, given an $R$-module homomorphism $\alpha\colon N \to N'$, there is a commutative diagram

$$\begin{array}{ccc} \operatorname{Hom}_R(L \oplus M, N) & \xrightarrow{\Phi} & \operatorname{Hom}_R(L, N) \oplus \operatorname{Hom}_R(M, N) \\ {\scriptstyle \operatorname{Hom}_R(L\oplus M,\alpha)}\downarrow & & \downarrow{\scriptstyle \operatorname{Hom}_R(L,\alpha)\oplus\operatorname{Hom}_R(M,\alpha)} \\ \operatorname{Hom}_R(L \oplus M, N') & \xrightarrow{\Phi'} & \operatorname{Hom}_R(L, N') \oplus \operatorname{Hom}_R(M, N') \end{array}$$

(c) (Bonus) Prove that, given a set of $R$-modules $\{M_i\}_{i\in I}$, one has

$$\operatorname{Hom}_R(\oplus_i M_i, N) \cong \prod_i \operatorname{Hom}_R(M_i, N).$$

**Exercise 2.** (a) Prove that given two exact sequences of unitary $R$-module homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0 \qquad\qquad 0 \to C \xrightarrow{\gamma} D \xrightarrow{\delta} E$$

the following sequence is also exact:

$$A \xrightarrow{\alpha} B \xrightarrow{\gamma\circ\beta} D \xrightarrow{\delta} E.$$

(b) Prove that given a unitary $R$-module $M$, there is an exact sequence

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} M \to 0$$

such that each $F_i$ is a free $R$-module.

(c) (Bonus) Prove that given a homomorphism of unitary $R$-modules $f\colon M \to M'$, there is a homomorphism of exact sequences

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\partial_2} & F_1 & \xrightarrow{\partial_1} & F_0 & \xrightarrow{\partial_0} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \widetilde{f}_1} & & \downarrow{\scriptstyle \widetilde{f}_0} & & \downarrow{\scriptstyle f} & & \\ \cdots & \xrightarrow{\partial_2'} & F_1' & \xrightarrow{\partial_1'} & F_0' & \xrightarrow{\partial_0'} & M' & \longrightarrow & 0 \end{array}$$

such that $F_i$ and $F_i'$ are free $R$-modules for all $i$.