MATH 720, Algebra I
Exercises 2
Due Wed 17 Sep

Exercise 1. Let G be an additive abelian group, and let $g_1, g_2, \ldots \in G$. Set $\mathbb{N}_+ = \{1, 2, \ldots\}$. For each $i \in \mathbb{N}_+$, let $\mathbf{e}_i = (0, \ldots, 0, 1, 0, \ldots) \in \mathbb{Z}^{(\mathbb{N}_+)}$ where the 1 occurs in the ith entry. For instance, we have $\mathbf{e}_1 = (1, 0, 0, 0, \ldots)$ and $\mathbf{e}_2 = (0, 1, 0, 0, \ldots)$. Prove that there is a unique homomorphism of additive abelian groups $f\colon \mathbb{Z}^{(\mathbb{N}_+)} \to G$ such that $f(\mathbf{e}_i) = g_i$ for all i.

Exercise 2. Let $f\colon R \to S$ be a homomorphism of rings, and let $I \subseteq R$ be an ideal. Define $\overline{f}\colon R/I \to S$ by the formula $\overline{f}(r + I) := f(r)$. Prove that $\overline{f}$ is a well-defined ring homomorphism if and only if $I \subseteq \operatorname{Ker}(f)$.

Exercise 3. Let R be a non-zero commutative ring with identity. Prove that R is a field if and only if the only ideals of R are 0 and R.