MATH 720, Algebra I
Exercises 3
Due Wed 24 Sep

Throughout this homework set, let R be a non-zero commutative ring with identity.

Exercise 1. Let k be a field, and let $f\colon k \to R$ be a homomorphism of commutative rings with identity. Prove that f is a monomorphism.

Exercise 2. Let p be a prime number, and assume that R is a $\mathbb{Z}_p$-algebra. Let $F\colon R \to R$ be defined by $F(r) = r^p$.

(a) Prove that F is a homomorphism of $\mathbb{Z}_p$-algebras. (Hint: You may assume that the Binomial Theorem holds in R, that is, for all $r, s \in R$ and all $n \in \mathbb{N}$, we have $(r+s)^n = \sum_{i=0}^{n} \binom{n}{i} r^i s^{n-i}$.)

(b) Prove that if R is an integral domain, then F is a monomorphism.

The map F is called the "Frobenius endomorphism" of R.

Exercise 3. Consider the polynomial ring $R[x]$.

(a) Let $f \in R[x]$ be monic. Prove that f is a unit in $R[x]$ if and only if $f = 1_R$.

(b) Give an example of a non-zero commutative ring R with identity having a non-constant polynomial $f \in R[x]$ that is a unit in $R[x]$. (In particular, the assumption that f is monic is necessary in part (a).)