MATH 720, Algebra I
Exercises 5
Due Wed 08 Oct

Exercise 1. Let R be an integral domain, and let $a, b, c \in R$ such that $a \neq 0$.

(a) Prove that if $\langle ab \rangle = \langle ac \rangle$, then $\langle b \rangle = \langle c \rangle$.
(b) Prove that if $\langle a \rangle = \langle ab \rangle$, then b is a unit.

Exercise 2. Let R be an integral domain, and consider the polynomial ring $S = R[x_1, \ldots, x_n]$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$, and let $f \in S$. Prove that there are polynomials $q_1, \ldots, q_n$ such that $q_i = q_i(x_1, \ldots, x_i) \in R[x_1, \ldots, x_i]$ for $i = 1, \ldots, n$ and $f = [\sum_i q_i \cdot (x_i - a_i)] + f(\mathbf{a})$. (Hint. Use induction on n.)

Exercise 3. Let R be an integral domain, and consider the polynomial ring $S = R[x_1, \ldots, x_n]$. Let $Z \subseteq S$, and set $I = (Z)S$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$, and consider the ideal $\mathfrak{m} := (x_1 - a_1, \ldots, x_n - a_n)S$. Prove that the following conditions are equivalent.

(i) $I \subseteq \mathfrak{m}$.
(ii) For all $f \in I$, we have $f(\mathbf{a}) = 0$.
(iii) For all $f \in Z$, we have $f(\mathbf{a}) = 0$.

Exercise 4. Consider the polynomial ring $\mathbb{R}[x]$ and the ideal $\langle x^2 + 1 \rangle \subseteq \mathbb{R}[x]$.

(i) Prove that there is an isomorphism of $\mathbb{R}$-algebras $\mathbb{C} \cong \mathbb{R}[x]/\langle x^2 + 1 \rangle$.
(ii) Prove that $\langle x^2 + 1 \rangle$ is a maximal ideal of $\mathbb{R}[x]$.