MATH 720, Algebra I
Exercises 6
Due Wed 15 Oct

Exercise 1. Let R be a non-zero commutative ring with identity, and let a, b be non-zero elements of R. Consider the following conditions.

(i) There is a unit $u \in R$ such that $b = ua$.
(ii) $a \mid b$ and $b \mid a$.
(iii) $\langle a \rangle = \langle b \rangle$.

(a) Prove that (i) $\implies$ (ii) $\iff$ (iii).
(b) Prove that if R is an integral domain, then conditions (i)–(iii) are equivalent.
(c) Give an example showing that the implication (ii) $\implies$ (i) may not hold if R is not an integral domain.

Exercise 2. Let $R = \mathbb{R}[x^2, x^3] \subseteq \mathbb{R}[x]$, as in Example 2.5.8 from the notes. Prove that the element x^2 is irreducible in R.

Exercise 3. Let R be an integral domain with field of fractions $Q(R)$, and let $a, b \in R$ such that $b \neq 0$. Prove that $a/b \in R$ if and only if $b \mid a$.

Exercise 4. Let R be an integral domain, and let $p, q \in R$ be prime elements. Prove that $p \mid q$ if and only if there is a unit u such that $q = up$.

Exercise 5. Let R be a unique factorization domain with $a, b, c \in R$. Prove that if $a \mid bc$ and $\gcd(a, b) = 1$, then $a \mid c$.