MATH 720, Algebra I
Exercises 7
Due Wed 22 Oct

Exercise 1. Let R be a UFD and $0 \neq f = \sum_{i=0}^{d} a_i x^i \in R[x]$.
(a) Show that $C(tf) = [t]C(f)$ for each $t \in R - \{0\}$.
(b) Show that if $C(f) = [r]$, then there is a primitive polynomial $g \in R[x]$ such that $f = rg$.

Exercise 2. Let R be an integral domain, and $0 \neq a, b \in R$.
(a) Prove that if $d \in R$ such that $\langle a, b\rangle = \langle d\rangle$, then $\gcd(a, b) = [d]$.
(b) Prove that if $m \in R$ such that $\langle a\rangle \bigcap \langle b\rangle = \langle m\rangle$, then $\operatorname{lcm}(a, b) = [m]$.
(c) (Bonus) Prove or disprove the converses of parts (a) and (b). If the converse fails in general, give an additional condition on R that makes each converse hold.

Exercise 3. Let R be an integral domain. Let $c_0, c_1, \ldots, c_n$ be distinct elements of R, and let $d_0, \ldots, d_n \in R$. Prove that there is at most one polynomial $f \in R[x]$ of degree n such that $f(c_i) = d_i$ for $i = 0, 1, \ldots, n$.

Exercise 4 (Lagrange interpolation)**.** Let k be a field. Let $c_0, c_1, \ldots, c_n$ be distinct elements of k, and let $d_0, \ldots, d_n \in k$. Prove that

$$f = \sum_{i=0}^{n} \frac{(x - c_0)\cdots(x - c_{i-1})(x - c_{i+1})\cdots(x - c_n)}{(c_i - c_0)\cdots(c_i - c_{i-1})(c_i - c_{i+1})\cdots(c_i - c_n)} d_i$$

is the unique polynomial in $k[x]$ of degree n such that $f(c_i) = d_i$ for $i = 0, 1, \ldots, n$.