MATH 720, Algebra I
Exercises 8
Due Wed 29 Oct

Exercise 1. Let $R = \{a + b\sqrt{10} \mid a, b \in \mathbb{Z}\} \subseteq \mathbb{R}$. Define $N\colon R \to \mathbb{Z}$ by the formula $N(a + b\sqrt{10}) := a^2 - 10b^2$. Let $u, v \in R$.

(a) Prove that R is a subring of $\mathbb{R}$ with identity $1_R = 1_{\mathbb{R}}$. Deduce that R is an integral domain.
(b) Prove that $N(uv) = N(u)N(v)$.
(c) Prove or disprove: N is a ring homomorphism.
(d) Prove that $N(u) = 0$ if and only if $u = 0$.
(e) Prove that u is a unit in R if and only if $N(u) = \pm 1$.
(f) Prove that $2, 3, 4 + \sqrt{10}, 4 - \sqrt{10}$ are irreducible elements of R.
(g) Prove that $2, 3, 4 + \sqrt{10}, 4 - \sqrt{10}$ are not associates in R.
(h) Show that $2 \cdot 3 = (4 + \sqrt{10})(4 - \sqrt{10})$.
(i) Prove that $2, 3, 4 + \sqrt{10}, 4 - \sqrt{10}$ are not prime elements of R.
(j) Prove that R is not a UFD.
(k) Prove or disprove: R is a Euclidean domain.
(l) Prove that every non-zero nonunit of R factors as a finite product of irreducible elements of R.