MATH 720, Algebra I
Exercises 9
Due Wed 12 Nov

Throughout this homework set, let R be a non-zero commutative ring with identity.

Exercise 1. Let M be an R-module, and fix a subset $S \subseteq \operatorname{Hom}_R(M, M)$. Set

$$M^S := \{m \in M \mid f(m) = m \text{ for all } f \in S\}.$$

Prove that M^S is a submodule of M.

Exercise 2. Let I be an ideal of R.

(a) Prove that if I is free as an R-module, then I is principal. (Hint: Let $X \subseteq R$ such that $|X| \geq 2$. Prove that X is not linearly independent over R.)

(b) Prove that every principal ideal of R is free if and only if R is an integral domain. (In other words, the converse to part (a) holds if and only if R is an integral domain.)

(c) Let k be a field and consider the polynomial ring $R = k[X, Y]$. Prove that the ideal $\langle X, Y \rangle \subseteq R$ is not free as an R-module. Is this ideal free over k? (Justify your answer.)

Exercise 3. Let M and N be R-modules.

(a) For all $f, g \in \operatorname{Hom}_R(M, N)$ define $f + g \colon M \to N$ by the formula $(f + g)(m) := f(m) + g(m)$. For all $r \in R$ and all $f \in \operatorname{Hom}_R(M, N)$ define $rf \colon M \to N$ by the formula $(rf)(m) := f(rm) = rf(m)$. Prove that this definition makes $\operatorname{Hom}_R(M, N)$ into an R-module.

(b) (Bonus) For $n \in \mathbb{N}$, show that the map $\Phi \colon \operatorname{Hom}_R(R^n, N) \to N^n$ given by $\Phi(f) = (f(e_1), \ldots, f(e_n))$ is an R-module isomorphism.