MATH 720, Algebra I
Midterm Exam
Due Wed 05 Nov

Instructions. You may use the course notes and your class notes on this exam, but you must work alone on it. (You can ask the instructor questions, but you may not work with other students, tutors, friends, instructors, etc., and you may not use any online resources.)

Exercise 1. Let $\phi\colon R \to S$ be a ring isomorphism, and let $z \in R$. Assume that R is a commutative ring with identity.

(a) Prove that S is a commutative ring with identity.
(b) Prove that R is an integral domain if and only if S is an integral domain.
(c) Prove that z is irreducible in R if and only if $\phi(z)$ is irreducible in S.

Exercise 2. Let R be a commutative ring with identity, and let $b, c \in R$ such that c is a unit. Prove that the function $\phi\colon R[x] \to R[x]$ given by $\phi(f) = f(cx + b)$ is an isomorphism of R-algebras.

Exercise 3. Let p be a (positive) prime integer.

(a) Prove that the polynomial $f = \sum_{i=0}^{p-1} x^i = (x^p - 1)/(x - 1)$ is irreducible in $\mathbb{Z}[x]$. Hint: Use the binomial theorem and Eisenstein's criterion to prove that $f(x + 1)$ is irreducible.
(b) Prove or disprove: The polynomial f from part (a) is prime in $\mathbb{Z}[x]$.

Exercise 4. Let R be a commutative ring with identity, let J be an ideal of R, and let S be a subset of R. Set $I = (S)R$

(a) Prove that the set $(J : S) := \{x \in R \mid xs \in J \text{ for all } s \in S\}$ is an ideal of R.
(b) Prove that $(J : S) = R$ if and only if $S \subseteq J$.
(c) Prove that $(J : S) = (J : I)$.