HOMOLOGICAL ALGEBRA, SPRING 2009
FINAL HOMEWORK (OPTIONAL)

Exercise 1. Let $(R, \mathfrak{m})$ be a commutative noetherian local ring, and let M and N be nonzero finitely generated R-modules. Show that, if $r = \mathrm{pd}_R(M) < \infty$, then $\mathrm{Ext}^r_R(M, N) \neq 0$. [Hint: Use the long exact sequence in $\mathrm{Ext}^i_R(M, -)$ associated to the short exact sequence $0 \to \mathfrak{m}N \to N \to N/\mathfrak{m}N \to 0$.]

Exercise 2. (Functoriality of long exact sequences) Let R be a commutative ring and consider the following diagram of chain maps:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_\bullet & \xrightarrow{F_\bullet} & M_\bullet & \xrightarrow{G_\bullet} & M''_\bullet & \longrightarrow & 0 \\
 & & {\scriptstyle f_\bullet}\downarrow & & {\scriptstyle g_\bullet}\downarrow & & {\scriptstyle h_\bullet}\downarrow & & \\
0 & \longrightarrow & N'_\bullet & \xrightarrow{H_\bullet} & N_\bullet & \xrightarrow{K_\bullet} & N''_\bullet & \longrightarrow & 0.
\end{array}$$

Assume that, for each integer i, the following diagram commutes:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & M'_i & \xrightarrow{F_i} & M_i & \xrightarrow{G_i} & M''_i & \longrightarrow & 0 \\
 & & {\scriptstyle f_i}\downarrow & & {\scriptstyle g_i}\downarrow & & {\scriptstyle h_i}\downarrow & & \\
0 & \longrightarrow & N'_i & \xrightarrow{H_i} & N_i & \xrightarrow{K_i} & N''_i & \longrightarrow & 0.
\end{array}$$

Show that the following diagram of long exact sequences commutes:

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{\eth^M_{i+1}} & \mathrm{H}_i(M'_\bullet) & \xrightarrow{\mathrm{H}_i(F_\bullet)} & \mathrm{H}_i(M_\bullet) & \xrightarrow{\mathrm{H}_i(G_\bullet)} & \mathrm{H}_i(M''_\bullet) & \xrightarrow{\eth^M_i} & \mathrm{H}_{i-1}(M'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(F_\bullet)} & \cdots \\
 & & {\scriptstyle \mathrm{H}_i(f_\bullet)}\downarrow & & {\scriptstyle \mathrm{H}_i(g_\bullet)}\downarrow & & {\scriptstyle \mathrm{H}_i(h_\bullet)}\downarrow & & {\scriptstyle \mathrm{H}_{i-1}(f_\bullet)}\downarrow & & \\
\cdots & \xrightarrow{\eth^N_{i+1}} & \mathrm{H}_i(N'_\bullet) & \xrightarrow{\mathrm{H}_i(H_\bullet)} & \mathrm{H}_i(N_\bullet) & \xrightarrow{\mathrm{H}_i(K_\bullet)} & \mathrm{H}_i(N''_\bullet) & \xrightarrow{\eth^N_i} & \mathrm{H}_{i-1}(N'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(H_\bullet)} & \cdots .
\end{array}$$