MATH 499/696, SPRING 2010, HOMEWORK 2
DUE FRIDAY 19 FEBRUARY

Item numbers refer to version of course notes dated February 5, 2010

Exercise 1.3.16(b): Verify the equality $(X+XY, Y+XY, X^2, Y^2)R = (X,Y)R$ of ideals in the polynomial ring $R = \mathbb{Q}[X,Y]$.

Exercise 1.3.18: Let A be a commutative ring with identity. Let $R = A[X,Y]$ and show that the ideal $(X,Y)R$ is not principal.

Exercise 1.4.28: Let A denote a commutative ring with identity, and set $R = A[X_1,\ldots,X_d]$ and $(\mathbf{X}) = (X_1,\ldots,X_d)R$. Show that, if $f \in R$, then $f \in (\mathbf{X})^n$ if and only if each monomial occurring in f has degree at least n. Show that this implies that $(\mathbf{X}) \neq R$.

Exercise 1.5.18: Verify the properties in Fact 1.5.11: Let R be a commutative ring with identity, let Let n be a positive integer, and let I and J be ideals of R.

(a) Suppose $I = (f_1,\ldots,f_m)R$. Then $\mathrm{rad}(I) \subseteq \mathrm{rad}(J)$ if and only if for each $i = 1,2,\ldots,m$ there exists a positive integer n_i such that $f_i^{n_i} \in J$.

(b) Suppose $I = (f_1,\ldots,f_s)R$ and $J = (g_1,\ldots,g_t)R$. Then $\mathrm{rad}(I) = \mathrm{rad}(J)$ if and only if for each $i = 1,2,\ldots,s$ there exists a positive integer n_i such that $f_i^{n_i} \in J$, and for each $j = 1,2,\ldots,t$ there exists a positive integer m_j such that $g_j^{m_j} \in I$.

(c) Suppose $I \subseteq J$ and that $J = (g_1,\ldots,g_t)R$. Then $\mathrm{rad}(I) = \mathrm{rad}(J)$ if and only if for each $j = 1,2,\ldots,t$ there exists an integer m_j such that $g_j^{m_j} \in I$.

Exercise 1.5.20: [Math 696 only] Let A be a commutative ring with identity. Consider the polynomial ring $R = A[X_1,\ldots,X_d]$ and the ideal $(\mathbf{X}) = (X_1,\ldots,X_d)R$.

(a) Prove that $((\mathbf{X})^n : (\mathbf{X})) = (\mathbf{X})^{n-1}$.

(b) List the monomials in $(\mathbf{X})^n \smallsetminus (\mathbf{X})^{n-1}$.