MATH 499/696, SPRING 2010, HOMEWORK 3
DUE FRIDAY 05 MARCH

Item numbers refer to version of course notes dated February 5, 2010

Exercise 2.1.19: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Set $(\mathbf{X}) = (X_1, \ldots, X_d)R$ and let I be a monomial ideal. Prove that $I \neq R$ if and only if $I \subseteq (\mathbf{X})$.

Exercise 2.1.20: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Set $(\mathbf{X}) = (X_1, \ldots, X_d)R$. Prove that, if I is a monomial ideal such that $I \neq R$, then $\operatorname{rad}(I) = \operatorname{rad}((\mathbf{X}))$ if and only if for each $i = 1, \ldots, d$ there exists an integer $n_i > 0$ such that $X_i^{n_i} \in I$.

Exercise 2.3.19: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $I_1, \ldots, I_n$ be monomial ideals in R and set $(\mathbf{X}) = (X_1, \ldots, X_d)R$.

(a) Prove that the product $I_1 \cdots I_n$ is a monomial ideal.
(b) Prove that, if $\operatorname{rad}(I_j) = \operatorname{rad}((\mathbf{X}))$ for $j = 1, \ldots, n$, then $\operatorname{rad}(I_1 \cdots I_n) = \operatorname{rad}((\mathbf{X}))$.
(c) Prove that if $I_j \neq R$ for $j = 1, \ldots, n$ and $\operatorname{rad}(I_1 \cdots I_n) = \operatorname{rad}((\mathbf{X}))$, then $\operatorname{rad}(I_j) = \operatorname{rad}((\mathbf{X}))$ for $j = 1, \ldots, n$.
(d) Prove that $[I_1 \cdots I_n] = \{z_1 \cdots z_n \mid z_1 \in [I_1], \ldots, z_n \in [I_n]\}$.

Exercise 2.3.20: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $I_1, \ldots, I_n$ be monomial ideals in R and set $(\mathbf{X}) = (X_1, \ldots, X_d)R$.

(a) Prove that the sum $I_1 + \cdots + I_n$ is a monomial ideal.
(b) Prove that, if $\operatorname{rad}(I_j) = \operatorname{rad}((\mathbf{X}))$ for $j = 1, \ldots, n$, then $\operatorname{rad}(I_1 + \cdots + I_n) = \operatorname{rad}((\mathbf{X}))$.
(c) Prove or give a counter-example for the following: if $\operatorname{rad}(I_1 + \cdots + I_n) = \operatorname{rad}((\mathbf{X}))$, then $\operatorname{rad}(I_j) = \operatorname{rad}((\mathbf{X}))$ for $j = 1, \ldots, n$.
(d) Prove that $\Gamma(I_1 + \cdots + I_n) = \Gamma(I_1) \cup \cdots \cup \Gamma(I_n)$.
(e) Prove that $[I_1 + \cdots + I_n] = [I_1] \cup \cdots \cup [I_n]$.

Exercise 2.1.29: [Math 696 only] Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let $(\mathbf{X}) = (X_1, \ldots, X_d)R$. Let I be a monomial ideal in R, and let S denote the set of monomials in $R \smallsetminus I$. Prove that S is a finite set if and only if $\operatorname{rad}(I) \supseteq \operatorname{rad}((\mathbf{X}))$.