MATH 499/696, SPRING 2010, HOMEWORK 4
DUE FRIDAY 26 MARCH

Item numbers refer to version of course notes dated March 2, 2010

Exercise 2.3.22: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_5]$ in 5 variables. Find the irredundant monomial generating sequence and compute $\nu_R(I)$ for the monomial ideal

$$I = (X_1X_2^2X_3^3, X_1X_3, X_2X_4, X_1^3X_2^2X_4X_5)R.$$

Exercise 2.3.24: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let I be a monomial ideal of R. Prove that $\mathrm{rad}(I) = \mathrm{rad}((\mathbf{X}))$ if and only if the irredundant monomial generating sequence for I contains a power of each variable.

Exercise 3.1.13: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Suppose I is generated by the set of monomials S and J is generated by the set of monomials T. Prove or disprove the following: The ideal $I \cap J$ is generated by the set of monomials $L = \{\mathrm{lcm}(f, g) \mid f \in S \text{ and } g \in T\}$.

Exercise 3.3.17: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y]$ in two variables. In this exercise you are asked to work through the proof of Theorem 3.3.10 with $I = (X, Y)R$ and $J = (X^3, X^2Y^2, Y^4)R$.

(a) Prove that the hypotheses of Theorem 3.3.10 are satisfied for this ideal J.
(b) Start with $z_1 = X^3$ and follow the proof to find an element $w_2 \in (J :_R I) \smallsetminus J$. Graph z_1, w_2, and J on the same set of coordinate axes.

Exercise 2.3.21: [Math 696 only] Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables.

(a) Let $I_1, \ldots, I_k$, and J be monomial ideals in R. Prove that $(I_1 + \cdots + I_k) \cap J = (I_1 \cap J) + \cdots + (I_k \cap J)$.
(b) Give an example (where $d = 2$) to show that this is not true without the assumption that each of the ideals I_1, I_2, and J are monomial ideals.