MATH 499/696, SPRING 2010, HOMEWORK 6
DUE FRIDAY 23 APRIL

Item numbers refer to version of course notes dated 08 April 2010

Exercises 4.1.13 and 4.1.11(a): Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a non-zero m-irreducible monomial ideal in R. Prove that there are positive integers $n, t_1, \ldots, t_n$ such that $\text{m-rad}\,(J) = (X_{t_1}, \ldots, X_{t_n})R$. Prove that $\text{m-rad}\,(J)$ is m-irreducible.

Exercise 3.5.27: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R such that $J \neq R$. Prove that the following conditions are equivalent:

(i) J is m-irreducible;
(ii) for all monomial ideal J_1, J_2 if $J_1 \cap J_2 \subseteq J$, then either $J_1 \subseteq J$ or $J_2 \subseteq J$; and
(iii) for all monomials $f, g \in [R]$ if $\text{lcm}(f, g) \in J$, then either $f \in J$ or $g \in J$.

Exercise 4.3.12(a): Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R with irredundant m-irreducible decomposition $J = \cap_{i=1}^n J_i$. Assume that for $i = 1, \ldots, n$ the ideal J_i is square-free. Prove that J is square-free.

Exercise 4.5.9: Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X, Y]$ in 2 variables. Find an irredundant m-irreducible decomposition of the ideal $J^{[3]}$ where $J = (X^3, X^2Y, Y^3)R$. Justify your answer.

Exercise 4.1.11(b): [Math 696 or Extra Credit] Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a non-zero monomial ideal in R. Prove or disprove the following: if $\text{m-rad}\,(J)$ is m-irreducible, then J is m-irreducible.

Exercise 4.1.14: [Math 696 or Extra Credit] Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a non-zero m-irreducible monomial ideal in R. Prove or disprove the following: If $\{I_\lambda\}_{\lambda \in \Lambda}$ is a (possibly infinite) set of monomial ideals in R such that $J = \cap_{\lambda \in \Lambda} I_\lambda$, then there is an index $\lambda \in \Lambda$ such that $J = I_\lambda$.

Exercise 4.3.12(b): [Math 696 or Extra Credit] Let A be a commutative ring with identity and let R be the polynomial ring $R = A[X_1, \ldots, X_d]$ in d variables. Let J be a monomial ideal in R with irredundant m-irreducible decomposition $J = \cap_{i=1}^n J_i$. Prove or disprove the following: If J is square-free, then each ideal J_i is square-free.