MATH 724, SPRING 2012, HOMEWORK 1
DUE WEDNESDAY 01 FEBRUARY

Assumptions: k is a field, and $R = k[X_1, \ldots, X_n]$ for some $n \geqslant 1$.

Exercise 1. Let $I \subseteq R$ be an ideal, and consider the "radical" of I:

$$\operatorname{rad}(I) = \{x \in R \mid x^n \in I \text{ for some } n \geqslant 0\}.$$

Note that $\operatorname{rad}(I)$ is an ideal of R containing I. Prove that $V(I) = V(\operatorname{rad}(I))$.

Exercise 2. Let $S, S' \subseteq R$.

(a) Prove that if $S \subseteq S'$, then $V(S) \supseteq V(S')$.

(b) Prove or give a counterexample to the following: if $V(S) \supseteq V(S')$, then $(S)R \subseteq (S')R$.

Exercise 3. For each $\underline{a} = (a_1, \ldots, a_n) \in \mathbb{A}^n_k$, set $\mathfrak{m}_{\underline{a}} = (X_1 - a_1, \ldots, X_n - a_n)R$. Let $S \subseteq R$, and prove that $\underline{a} \in V(S)$ if and only if $\mathfrak{m}_{\underline{a}} \supseteq S$.

Exercise 4. Let $m \geqslant 1$. A function $F\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ is *regular* if there are polynomials $f_1, \ldots, f_m \in R$ such that $F(\underline{a}) = (f_1(\underline{a}), \ldots, f_m(\underline{a}))$ for all $\underline{a} \in \mathbb{A}^n_k$.

(a) Prove that every regular function $F\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ is continuous.

(b) Prove that if $F\colon \mathbb{A}^n_k \to \mathbb{A}^m_k$ and $G\colon \mathbb{A}^p_k \to \mathbb{A}^n_k$ are regular, then so is the composition $F \circ G\colon \mathbb{A}^p_k \to \mathbb{A}^m_k$.