MATH 724, SPRING 2012, HOMEWORK 4
DUE WEDNESDAY 21 MARCH

Exercise 1. Find the irreducible components for the following algebraic sets.
(a) $V(X+Y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$.
(b) $V(X^2+Y^2) \subseteq \mathbb{A}^2_{\mathbb{C}}$.
(c) $V(XY, XZ) \subseteq \mathbb{A}^3_{\mathbb{C}}$.
Justify your answers.

Exercise 2. Let $R = k[X_0, \ldots, X_n]$. For each $d \in \mathbb{Z}$, set

$$k(R)_d = \{f/g \in k(X_0, \ldots, X_n) \mid f \in R_i \text{ and } 0 \neq g \in R_{i-d} \text{ for some } i \geqslant d\}.$$

For each nonzero element $h \in R_j$ and each $d \in \mathbb{Z}$, set

$$(R_h)_d = \{f/h^m \in k(X_0, \ldots, X_n) \mid f \in R_{d+mj} \text{ and } m \geqslant 0\}.$$

(a) Show that $k(R)_0 \subseteq \oplus_{d \in \mathbb{Z}} k(R)_d \subseteq k(X_0, \ldots, X_n)$ are subrings and that $k(R)_0 \subseteq k(X_0, \ldots, X_d)$ is a subfield.
(b) Show that $(R_h)_0 \subseteq \oplus_{d \in \mathbb{Z}} (R_h)_d \subseteq k(X_0, \ldots, X_n)$ are subrings.
(c) Show that there is a field isomorphism $k(R)_0 \cong k(Y_1, \ldots, Y_n)$ where $Y_1, \ldots, Y_n$ is a sequence of indeterminates.