MATH 724, SPRING 2012, HOMEWORK 1
DUE FRIDAY 27 JAN

Let R be a commutative ring with identity.

Exercise 1. Let $\{f_i\colon M_i \to N_i\}_{i\in I}$ be a set of R-module homomorphisms.

(a) Prove that there are well-defined R-module homomorphisms

$$\prod_{i\in I} f_i\colon \prod_{i\in I} M_i \to \prod_{i\in I} N_i \qquad \text{and} \qquad \coprod_{i\in I} f_i\colon \coprod_{i\in I} M_i \to \coprod_{i\in I} N_i$$

given in each case by the rule $(m_i)_i \mapsto (f_i(m_i))_i$.

(b) Prove that $\operatorname{Ker}(\prod_{i\in I} f_i) = \prod_{i\in I}\operatorname{Ker}(f_i)$ and $\operatorname{Ker}(\coprod_{i\in I} f_i) = \coprod_{i\in I}\operatorname{Ker}(f_i)$.

(c) Prove that $\operatorname{Im}(\prod_{i\in I} f_i) = \prod_{i\in I}\operatorname{Im}(f_i)$ and $\operatorname{Im}(\coprod_{i\in I} f_i) = \coprod_{i\in I}\operatorname{Im}(f_i)$.

(d) Prove that $\operatorname{Coker}(\prod_{i\in I} f_i) \cong \prod_{i\in I}\operatorname{Coker}(f_i)$ and $\operatorname{Coker}(\coprod_{i\in I} f_i) \cong \coprod_{i\in I}\operatorname{Coker}(f_i)$.

(e) Given another set $\{g_i\colon N_i \to P_i\}_{i\in I}$ of R-module homomorphisms, prove that the following conditions are equivalent:

(i) The sequence $\prod_{i\in I} M_i \xrightarrow{\prod_{i\in I} f_i} \prod_{i\in I} N_i \xrightarrow{\prod_{i\in I} g_i} \prod_{i\in I} P_i$ is exact.

(ii) The sequence $\coprod_{i\in I} M_i \xrightarrow{\coprod_{i\in I} f_i} \coprod_{i\in I} N_i \xrightarrow{\coprod_{i\in I} g_i} \coprod_{i\in I} P_i$ is exact.

(iii) For each $i \in I$, the sequence $M_i \xrightarrow{f_i} N_i \xrightarrow{g_i} P_i$ is exact.

Exercise 2. Let M be an R-module, and prove that the following conditions are equivalent:

(i) $M = 0$.

(ii) For each multiplicatively closed subset $U \subseteq R$, one has $U^{-1}M = 0$.

(iii) For each prime ideal $\mathfrak{p} \subset R$, one has $M_{\mathfrak{p}} = 0$.

(iv) For each maximal ideal $\mathfrak{m} \subset R$, one has $M_{\mathfrak{m}} = 0$.