MATH 726, SPRING 2012, HOMEWORK 12
DUE FRIDAY 04 MAY

Let R be a commutative ring with identity.

Exercise 1. (Morphisms of long exact sequences) Consider the following diagram of chain maps with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_\bullet & \xrightarrow{F_\bullet} & M_\bullet & \xrightarrow{G_\bullet} & M''_\bullet & \longrightarrow & 0 \\ & & {\scriptstyle f_\bullet}\big\downarrow & & {\scriptstyle g_\bullet}\big\downarrow & & {\scriptstyle h_\bullet}\big\downarrow & & \\ 0 & \longrightarrow & N'_\bullet & \xrightarrow{H_\bullet} & N_\bullet & \xrightarrow{K_\bullet} & N''_\bullet & \longrightarrow & 0. \end{array}$$

Assume that, for each integer i, the following diagram commutes:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M'_i & \xrightarrow{F_i} & M_i & \xrightarrow{G_i} & M''_i & \longrightarrow & 0 \\ & & {\scriptstyle f_i}\big\downarrow & & {\scriptstyle g_i}\big\downarrow & & {\scriptstyle h_i}\big\downarrow & & \\ 0 & \longrightarrow & N'_i & \xrightarrow{H_i} & N_i & \xrightarrow{K_i} & N''_i & \longrightarrow & 0. \end{array}$$

Show that the following diagram of long exact sequences commutes:

$$\begin{array}{ccccccccccc} \cdots & \xrightarrow{\eth^M_{i+1}} & \mathrm{H}_i(M'_\bullet) & \xrightarrow{\mathrm{H}_i(F_\bullet)} & \mathrm{H}_i(M_\bullet) & \xrightarrow{\mathrm{H}_i(G_\bullet)} & \mathrm{H}_i(M''_\bullet) & \xrightarrow{\eth^M_i} & \mathrm{H}_{i-1}(M'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(F_\bullet)} & \cdots \\ & & {\scriptstyle \mathrm{H}_i(f_\bullet)}\big\downarrow & & {\scriptstyle \mathrm{H}_i(g_\bullet)}\big\downarrow & & {\scriptstyle \mathrm{H}_i(h_\bullet)}\big\downarrow & & {\scriptstyle \mathrm{H}_{i-1}(f_\bullet)}\big\downarrow & & \\ \cdots & \xrightarrow{\eth^N_{i+1}} & \mathrm{H}_i(N'_\bullet) & \xrightarrow{\mathrm{H}_i(H_\bullet)} & \mathrm{H}_i(N_\bullet) & \xrightarrow{\mathrm{H}_i(K_\bullet)} & \mathrm{H}_i(N''_\bullet) & \xrightarrow{\eth^N_i} & \mathrm{H}_{i-1}(N'_\bullet) & \xrightarrow{\mathrm{H}_{i-1}(H_\bullet)} & \cdots. \end{array}$$

Exercise 2. Let $F_\bullet \colon X_\bullet \to Y_\bullet$ be a chain map, and let M be an R-module. Prove that there is an isomorphism

$$\mathrm{Cone}(\mathrm{Hom}_R(F_\bullet, M)) \cong \Sigma\,\mathrm{Hom}_R(\mathrm{Cone}(F_\bullet), M).$$