MATH 726, SPRING 2012, HOMEWORK 4
DUE FRIDAY 17 FEBRUARY

Exercise 1. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings with identity. Let M be an R-module, and let N be an S-module. Let $X \subseteq M$ and $Y \subseteq N$ be subsets.

(a) Prove that if M is generated over R by X and N is generated over S by Y, then $N \otimes_R M$ is generated over S by $Z = \{y \otimes x \in N \otimes M \mid x \in X \text{ and } y \in Y\}$. In particular, if M is finitely generated over R and N is finitely generated over S, then $N \otimes_R M$ is finitely generated over S.

(b) Prove that if M is generated over R by X, then $S \otimes_R M$ is generated over S by $X' = \{1 \otimes x \in N \otimes M \mid x \in X\}$. In particular, if M is finitely generated over R, then $S \otimes_R M$ is finitely generated over S.

Exercise 2. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings with identity such that S is a flat R-module. Let M and N be R-modules such that M is finitely presented over R, that is, there is an exact sequence $R^m \xrightarrow{f} R^n \xrightarrow{g} M \to 0$ where m and n are non-negative integers. Prove that there is an S-module isomorphism $S \otimes_R \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_S(S \otimes_R M, S \otimes_R N)$

Exercise 3. Let M and N be flat R-modules. Prove that $M \otimes_R N$ is a flat R-module.

Exercise 4. Let $\varphi\colon R \to S$ be a homomorphism of commutative rings with identity. Prove that if M is a flat R-module, then $S \otimes_R M$ is a flat S-module.