MATH 726, SPRING 2012, HOMEWORK 7
DUE FRIDAY 09 MARCH

Let R be a commutative ring with identity. The Exercise below provides an example (due to Hyman Bass) of a non-projective $\mathbb{Z}$-modules M such that $M_\mathfrak{m}$ is projective for all maximal ideals $\mathfrak{m} \subset \mathbb{Z}$. The item marked * will be proved in class.

Exercise 1. Let $M \subseteq \mathbb{Q}$ be the $\mathbb{Z}$-submodule generated by the set $\{1/p \mid p \text{ is prime}\}$.

(a) Prove that $M = \{m/n \in \mathbb{Q} \mid n \text{ is either 1 or a product of distinct primes}\}$.
(b) Prove that M is not cyclic.
(c) Prove that for each maximal ideal $\mathfrak{m} = q\mathbb{Z} \subset \mathbb{Z}$, the localization $M_\mathfrak{m}$ is generated over $\mathbb{Z}_\mathfrak{m}$ by $1/q$.
(d) Prove that for each maximal ideal $\mathfrak{m} = q\mathbb{Z} \subset \mathbb{Z}$, there is a $\mathbb{Z}_\mathfrak{m}$-module isomorphism $M_\mathfrak{m} \cong \mathbb{Z}_\mathfrak{m}$.
(e) Prove that any subset $S \subseteq M$ with at least two elements must be linear dependent over $\mathbb{Z}$.
(f) Prove that M is not free as a $\mathbb{Z}$-module.
(g) * Every projective $\mathbb{Z}$-module is free.
(h) Prove that M is not projective as a $\mathbb{Z}$-module.