A somewhat gentle introduction to differential graded commutative algebra

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Abstract Differential graded (DG) commutative algebra provides powerful techniques for proving theorems about modules over commutative rings. These notes are a somewhat colloquial introduction to these techniques. In order to provide some motivation for commutative algebraists who are wondering about the benefits of learning and using these techniques, we present them in the context of a recent result of Nasseh and Sather-Wagstaff. These notes were used for the course "Differential Graded Commutative Algebra" that was part of the Workshop on Connections Between Algebra and Geometry at the University of Regina, May 29–June 1, 2012.

Dedicated with much respect to Tony Geramita

1 Introduction

Convention 1.1 The term "ring" is short for "commutative noetherian ring with identity", and "module" is short for "unital module". Let R be a ring.

These are notes for the course "Differential Graded Commutative Algebra" that was part of the Workshop on Connections Between Algebra and Geometry held at the University of Regina, May 29–June 1, 2012. They represent our attempt to provide a small amount of (1) motivation for commutative algebraists who are wondering about the benefits of learning and using Differential Graded (DG) techniques, and (2) actual DG techniques.

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DG Algebra

DG commutative algebra provides a useful framework for proving theorems about rings and modules, the statements of which have no reference to the DG universe. For instance, a standard theorem says the following:

Theorem 1.2 ([20, Corollary 1]) *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism, that is, a ring homomorphism making S into a flat R-module such that $\mathfrak{m}S \subseteq \mathfrak{n}$. Then S is Gorenstein if and only if R and $S/\mathfrak{m}S$ are Gorenstein. Moreover, there is an equality of Bass series $I_S(t) = I_R(t)I_{S/\mathfrak{m}S}(t)$.*

(See Definition 9.2 for the term “Bass series”.) Of course, the flat hypothesis is very important here. On the other hand, the use of DG algebras allows for a slight (or vast, depending on your perspective) improvement of this:

Theorem 1.3 ([9, Theorem A]) *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring homomorphism of finite flat dimension, that is, a local ring homomorphism such that S has a bounded resolution by flat R-module. Then there is a formal Laurent series $I_\varphi(t)$ with non-negative integer coefficients such that $I_S(t) = I_R(t)I_\varphi(t)$. In particular, if S is Gorenstein, then so is R.*

In this result, the series $I_\varphi(t)$ is the *Bass series* of φ. It is the Bass series of the “homotopy closed fibre” of φ (instead of the usual closed fibre $S/\mathfrak{m}S$ of φ that is used in Theorem 1.2) which is the commutative DG algebra $S \otimes^{\mathbf{L}}_R R/\mathfrak{m}$. In particular, when S is flat over R, this is the usual closed fibre $S/\mathfrak{m}S \cong S \otimes_R R/\mathfrak{m}$, so one recovers Theorem 1.2 as a corollary of Theorem 1.3.

Furthermore, DG algebra comes equipped with constructions that can be used to replace your given ring with one that is nicer in some sense. To see how this works, consider the following strategy for using completions.

To prove a theorem about a given local ring R, first show that the assumptions ascend to the completion $\widehat{R}$, prove the result for the complete ring $\widehat{R}$, and show how the conclusion for $\widehat{R}$ implies the desired conclusion for R. This technique is useful since frequently $\widehat{R}$ is nicer then R. For instance, $\widehat{R}$ is a homomorphic image of a power series ring over a field or a complete discrete valuation ring, so it is universally catenary (whatever that means) and it has a dualizing complex (whatever that is), while the original ring R may not have either of these properties.

When R is Cohen-Macaulay and local, a similar strategy sometimes allows one to mod out by a maximal R-regular sequence $\mathbf{x}$ to assume that R is artinian. The regular sequence assumption is like the flat condition for $\widehat{R}$ in that it (sometimes) allows for the transfer of hypotheses and conclusions between R and the quotient $\overline{R} := R/(\mathbf{x})$. The artinian hypothesis is particularly nice, for instance, when R contains a field because then $\overline{R}$ is a finite dimensional algebra over a field.

The DG universe contains a construction $\widetilde{R}$ that is similar $\overline{R}$, with an advantage and a disadvantage. The advantage is that it is more flexible than $\overline{R}$ because it does not require the ring to be Cohen-Macaulay, and it produces a finite dimensional algebra over a field regardless of whether or not R contains a field. The disadvantage is that $\widetilde{R}$ is a DG algebra instead of just an algebra, so it is graded commutative (almost, but not quite, commutative) and there is a bit more data to track when working with $\widetilde{R}$. However, the advantages outweigh the disadvantages in that $\widetilde{R}$ allows us to prove results for arbitrary local rings that can only be proved (as we understand things today) in special cases using $\overline{R}$. One such result is the following:

Theorem 1.4 ([32, Theorem A]) *A local ring has only finitely many semidualizing modules up to isomorphism.*

Even if you don't know what a semidualizing module is, you can see the point. Without DG techniques, we only know how to prove this result for Cohen-Macaulay rings that contain a field; see Theorem 2.13. With DG techniques, you get the unqualified result, which answers a question of Vasconcelos [41].

What These Notes Are

Essentially, these notes contain a sketch of the proof of Theorem 1.4; see 5.32, 7.38, and 8.17 below. Along the way, we provide a big-picture view of some of the tools and techniques in DG algebra (and other areas) needed to get a basic understanding of this proof. Also, since our motivation comes from the study of semidualizing modules, we provide a bit of motivation for the study of those gadgets in Appendix 9. In particular, we do not assume that the reader is familiar with the semidualizing world.

Since these notes are based on a course, they contain many exercises; sketches of solutions are contained in Appendix 10. They also contain a number of examples and facts that are presented without proof. A diligent reader may also wish to consider many of these as exercises.

What These Notes Are Not

These notes do not contain a great number of details about the tools and techniques in DG algebra. There are already excellent sources available for this, particularly, the seminal works [4, 6, 10]. The interested reader is encouraged to dig into these sources for proofs and constructions not given here. Our goal is to give some idea of what the tools look like and how they get used to solve problems. (To help readers in their digging, we provide many references for properties that we use.)

Notation

When it is convenient, we use notation from [11, 31]. Here we specify our conventions for some notions that have several notations:

$\mathrm{pd}_R(M)$: projective dimension of an R-module M
$\mathrm{id}_R(M)$: injective dimension of an R-module M
$\mathrm{len}_R(M)$: length of an R-module M
S_n: the symmetric group on $\{1,\ldots,n\}$.
$\mathrm{sgn}(\iota)$: the signum of an element $\iota \in S_n$.

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2 Semidualizing Modules

This section contains background material on semidualizing modules. It also contains a special case of Theorem 1.4; see Theorem 2.13. Further survey material can be found in [35, 38] and Appendix 9.

Definition 2.1 A finitely generated R-module C is *semidualizing* if the natural homothety map $\chi_C^R \colon R \to \mathrm{Hom}_R(C,C)$ given by $r \mapsto [c \mapsto rc]$ is an isomorphism and $\mathrm{Ext}_R^i(C,C) = 0$ for all $i \geqslant 1$. A *dualizing* R-module is a semidualizing R-module such that $\mathrm{id}_R(C) < \infty$. The set of isomorphism classes of semidualizing R-modules is denoted $\mathfrak{S}_0(R)$.

Remark 2.2 The symbol $\mathfrak{S}$ is an S, as in `\mathfrak{S}`.

Example 2.3 The free R-module R^1 is semidualizing.

Fact 2.4 The ring R has a dualizing module if and only if it is Cohen-Macaulay and a homomorphic image of a Gorenstein ring; when these conditions are satisfied, a dualizing R-module is the same as a "canonical" R-module. See Foxby [19, Theorem 4.1], Reiten [34, (3) Theorem], and Sharp [40, 2.1 Theorem (i)].

Remark 2.5 To the best of our knowledge, semidualizing modules were first introduced by Foxby [19]. They have been rediscovered independently by several authors who seem to all use different terminology for them. A few examples of this, presented chronologically, are:

Author(s)	Terminology	Context
Foxby [19]	"PG-module of rank 1"	commutative algebra
Vasconcelos [41]	"spherical module"	commutative algebra
Golod [23]	"suitable module" [1]	commutative algebra
Wakamatsu [43]	"generalized tilting module"	representation theory
Christensen [14]	"semidualizing module"	commutative algebra
Mantese and Reiten [30]	"Wakamatsu tilting module"	representation theory

The following facts are quite useful in practice.

Fact 2.6 Assume that $(R,\mathfrak{m})$ is local, and let C be a semidualizing R-module. If R is Gorenstein, then $C \cong R$. The converse holds if C is a dualizing R-module. See [14, (8.6) Corollary].

If $\mathrm{pd}_R(C) < \infty$, then $C \cong R$ by [38, Fact 1.14]. Here is a sketch of the proof. The isomorphism $\mathrm{Hom}_R(C,C) \cong R$ implies that $\mathrm{Supp}_R(C) = \mathrm{Spec}(R)$ and $\mathrm{Ass}_R(C) = \mathrm{Ass}_R(R)$. In particular, an element $x \in \mathfrak{m}$ is C-regular if and only if it is R-regular. If x is R-regular, it follows that C/xC is semidualizing over R/xR. By induction on $\mathrm{depth}(R)$, we conclude that $\mathrm{depth}_R(C) = \mathrm{depth}(R)$. The Auslander-Buchsbaum formula implies that C is projective, so it is free since R is local. Finally, the isomorphism $\mathrm{Hom}_R(C,C) \cong R$ implies that C is free of rank 1, that is, $C \cong R$.

Fact 2.7 Let $\varphi\colon R \to S$ be a ring homomorphism of finite flat dimension. (For example, this is so if φ is flat or surjective with kernel generated by an R-regular sequence.) If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module. The converse holds when φ is faithfully flat or local. The functor $S \otimes_R -$ induces a well-defined function $\mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ which is injective when φ is local. See [21, Theorems 4.5 and 4.9].

Exercise 2.8 Verify the conclusions of Fact 2.7 when φ is flat. That is, let $\varphi\colon R \to S$ be a flat ring homomorphism. Prove that if C is a semidualizing R-module, then the base-changed module $S \otimes_R C$ is a semidualizing S-module. Prove that the converse holds when φ is faithfully flat, e.g., when φ is local.

The next lemma is for use in Theorem 2.13, which is a special case of Theorem 1.4. See Remark 2.10 and Question 2.11 for further perspective.

Lemma 2.9 *Assume that R is local and artinian. Then there is an integer ρ depending only on R such that $\mathrm{len}_R(C) \leqslant \rho$ for every semidualizing R-module C.*

Proof. Let k denote the residue field of R. We show that the integer $\rho = \mathrm{len}_R(R)\mu^0_R$ satisfies the conclusion where $\mu^0_R = \mathrm{rank}_k(\mathrm{Hom}_R(k,R))$. (This is the 0th Bass number of R; see Definition 9.2.) Let C be a semidualizing R-module. Set $\beta = \mathrm{rank}_k(k \otimes_R C)$ and $\mu = \mathrm{rank}_k(\mathrm{Hom}_R(k,C))$. Since R is artinian and C is finitely generated, it follows that $\mu \geqslant 1$. Also, the fact that R is local implies that there is an R-module epimorphism $R^\beta \to C$, so we have $\mathrm{len}_R(C) \leqslant \mathrm{len}_R(R)\beta$. Thus, it remains to show that $\beta \leqslant \mu^0_R$.

The next sequence of isomorphisms uses adjointness and tensor cancellation:

$$
\begin{aligned}
k^{\mu_R^0} &\cong \operatorname{Hom}_R(k,R) \\
&\cong \operatorname{Hom}_R(k,\operatorname{Hom}_R(C,C)) \\
&\cong \operatorname{Hom}_R(C\otimes_R k,C) \\
&\cong \operatorname{Hom}_R(k\otimes_k(C\otimes_R k),C) \\
&\cong \operatorname{Hom}_k(C\otimes_R k,\operatorname{Hom}_R(k,C)) \\
&\cong \operatorname{Hom}_k(k^\beta,k^\mu) \\
&\cong k^{\beta\mu}.
\end{aligned}
$$

Since $\mu \geqslant 1$, it follows that $\beta \leqslant \beta\mu = \mu_R^0$, as desired. □

Remark 2.10 Assume that R is local and Cohen-Macaulay. If D is dualizing for R, then there is an equality $e_R(D) = e(R)$ of Hilbert-Samuel multiplicites with respect to the maximal ideal of R. See, e.g., [11, Proposition 3.2.12.e.i and Corollary 4.7.8]. It is unknown whether the same equality holds for an arbitrary *semi*dualizing R-module. Using the "additivity formula" for multiplicities, this boils down to the following. Some progress is contained in [17].

Question 2.11 Assume that R is local and artinian. For every semidualizing R-module C, must one have $\operatorname{len}_R(C) = \operatorname{len}(R)$?

While we are in the mood for questions, here is a big one. In every explicit calculation of $\mathfrak{S}_0(R)$, the answer is "yes"; see [36, 39].

Question 2.12 Assume that R is local. Must $|\mathfrak{S}_0(R)|$ be 2^n for some $n \in \mathbb{N}$?

Next, we sketch the proof of Theorem 1.4 when R is Cohen-Macaulay and contains a field. This sketch serves to guide the proof of the result in general.

Theorem 2.13 ([15, Theorem 1]) *Assume that $(R,\mathfrak{m},k)$ is Cohen-Macaulay local and contains a field. Then $|\mathfrak{S}_0(R)| < \infty$.*

Proof. Case 1: R is artinian, and k is algebraically closed. In this case, Cohen's structure theorem implies that R is a finite dimensional k-algebra. Since k is algebraically closed, a result of Happel [25, proof of first proposition in section 3] says that for each $n \in \mathbb{N}$ the following set is finite.

$$T_n = \{\text{isomorphism classes of } R\text{-modules } N \mid \operatorname{Ext}_R^1(N,N) = 0 \text{ and } \operatorname{len}_R(N) = n\}$$

Lemma 2.9 implies that there is a $\rho \in \mathbb{N}$ such that $\mathfrak{S}_0(R)$ is contained in the finite set $\bigcup_{n=1}^{\rho} T_n$, so $\mathfrak{S}_0(R)$ is finite.

Case 2: k is algebraically closed. Let $\mathbf{x} = x_1,\ldots,x_n \in \mathfrak{m}$ be a maximal R-regular sequence. Since R is Cohen-Macaulay, the quotient $R' = R/(\mathbf{x})$ is artinian. Also, R' has the same residue field as R, so Case 1 implies that $\mathfrak{S}_0(R')$ is finite. Since R is local, Fact 2.7 provides an injection $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$, so $\mathfrak{S}_0(R)$ is finite as well.

Case 3: the general case. A result of Grothendieck [24, Théorèm 19.8.2(ii)] provides a flat local ring homomorphism $R \to \overline{R}$ such that $\overline{R}/\mathfrak{m}\overline{R}$ is algebraically closed.

In particular, since R and $\overline{R}/\mathfrak{m}\overline{R}$ are Cohen-Macaulay, it follows that $\overline{R}$ is Cohen-Macaulay. The fact that R contains a field implies that $\overline{R}$ also contains a field. Hence, Case 2 shows that $\mathfrak{S}_0(\overline{R})$ is finite. Since R is local, Fact 2.7 provides an injective function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$, so $\mathfrak{S}_0(R)$ is finite as well. $\square$

Remark 2.14 Happel's result uses some deep ideas from algebraic geometry and representation theory. The essential point comes from a theorem of Voigt [42] (see also Gabriel [22, 1.2 Corollary]). We'll need a souped-up version of this result for the full proof of Theorem 1.4. This is the point of Section 8.

Remark 2.15 The proof of Theorem 2.13 uses the extra assumptions (extra compared to Theorem 1.4) in crucial places. The Cohen-Macaulay assumption is used in the reduction to the artinian case. And the fact that R contains a field is used in order to invoke Happel's result. In order to remove these assumptions for the proof of Theorem 1.4, we find an algebra U that is finite dimensional over an algebraically closed field such that there is an injective function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(U)$. The trick is that U is a DG algebra, and $\mathfrak{S}(U)$ is a set of equivalence classes of semidualizing DG U-modules. So, we need to understand the following:

(a) What are DG algebras, and how is U constructed?
(b) What are semidualizing DG modules, and how is the map $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(U)$ constructed?
(c) Why is $\mathfrak{S}(U)$ finite?

This is the point of the rest of the notes. See Sections 5, 7, and 8.

3 Hom Complexes

This section has several purposes. First, we set some notation and terminology. Second, we make sure that the reader is familiar with some notions that we need later in the notes. One of the main points of this section is Fact 3.18.

Complexes

The following gadgets form the foundation for homological algebra, and we shall use them extensively.

Definition 3.1 An *R-complex*[2] is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial_{i+1}^X} X_i \xrightarrow{\partial_i^X} X_{i-1} \xrightarrow{\partial_{i-1}^X} \cdots$$

[2] Readers more comfortable with notations like $X_\bullet$ or X_* for complexes should feel free to decorate their complexes as they see fit.

such that $\partial_i^X \partial_{i+1}^X = 0$ for all i. For each $x \in X_i$, the *degree* of x is $|x| := i$. The *ith homology module* of X is $\mathrm{H}_i(X) := \mathrm{Ker}(\partial_i^X)/\mathrm{Im}(\partial_{i+1}^X)$. A *cycle* in X_i is an element of $\mathrm{Ker}(\partial_i^X)$.

We use the following notation for augmented resolutions in several places below.

Example 3.2 Let M be an R-module. We consider M as an R-complex "concentrated in degree 0":

$$M = \quad 0 \to M \to 0.$$

Given an augmented projective resolution

$$P^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \to 0$$

the truncated resolution

$$P = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \to 0$$

is an R-complex such that $\mathrm{H}_0(P) \cong M$ and $\mathrm{H}_i(P) = 0$ for all $i \neq 0$. Similarly, given an augmented injective resolution

$${}^+I = \quad 0 \to M \xrightarrow{\varepsilon} I_0 \xrightarrow{\partial_0^I} I_{-1} \xrightarrow{\partial_{-1}^I} \cdots$$

the truncated resolution

$$I = \quad 0 \to I_0 \xrightarrow{\partial_0^I} I_{-1} \xrightarrow{\partial_{-1}^I} \cdots$$

is an R-complex such that $\mathrm{H}_0(I) \cong M$ and $\mathrm{H}_i(I) = 0$ for all $i \neq 0$.

The Hom Complex

The next constructions are used extensively in these notes. For instance, the chain maps are the morphisms in the category of R-complexes.

Definition 3.3 Let X and Y be R-complexes. The *Hom complex* $\mathrm{Hom}_R(X,Y)$ is defined as follows. For each integer n, set $\mathrm{Hom}_R(X,Y)_n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X_p, Y_{p+n})$ and $\partial_n^{\mathrm{Hom}_R(X,Y)}(\{f_p\}) := \{\partial_{p+n}^Y f_p - (-1)^n f_{p-1}\partial_p^X\}$. A *chain map* $X \to Y$ is a cycle in $\mathrm{Hom}_R(X,Y)_0$, i.e., an element of $\mathrm{Ker}(\partial_0^{\mathrm{Hom}_R(X,Y)})$. An element in $\mathrm{Hom}_R(X,Y)_0$ is *null-homotopic* if it is in $\mathrm{Im}(\partial_1^{\mathrm{Hom}_R(X,Y)})$. An *isomorphism* $X \xrightarrow{\cong} Y$ is a chain map $X \to Y$ with a two-sided inverse. We sometimes write f in place of $\{f_p\}$.

Exercise 3.4 Let X and Y be R-complexes.

(a) Prove that $\mathrm{Hom}_R(X,Y)$ is an R-complex.

(b) Prove that a chain map $X \to Y$ is a sequence of R-module homomorphisms $\{f_p \colon X_p \to Y_p\}$ making the following diagram commute:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial^X_{i+1}} & X_i & \xrightarrow{\partial^X_i} & X_{i-1} & \xrightarrow{\partial^X_{i-1}} & \cdots \\ & & \downarrow{\scriptstyle f_i} & & \downarrow{\scriptstyle f_{i-1}} & & \\ \cdots & \xrightarrow{\partial^Y_{i+1}} & Y_i & \xrightarrow{\partial^Y_i} & Y_{i-1} & \xrightarrow{\partial^Y_{i-1}} & \cdots . \end{array}$$

(c) Prove that if $\{f_p\} \in \operatorname{Hom}_R(X,Y)_0$ is null-homotopic, then it is a chain map.
(d) Prove that a sequence $\{f_p\} \in \operatorname{Hom}_R(X,Y)_0$ is null-homotopic if and only if there is a sequence $\{s_p \colon X_p \to Y_{p+1}\}$ of R-module homomorphisms such that $f_p = \partial^Y_{p+1} s_p + s_{p-1}\partial^X_p$ for all $p \in \mathbb{Z}$.

The following exercises contain useful properties of these constructions.

Exercise 3.5 ("Hom cancellation") Let X be an R-complex. Prove that the map $\tau\colon \operatorname{Hom}_R(R,X) \to X$ given by $\tau_n(\{f_p\}) = f_n(1)$ is an isomorphism of R-complexes.

Exercise 3.6 Let X be an R-complex, and let M be an R-module.

(a) Prove that $\operatorname{Hom}_R(M,X)$ is isomorphic to the following complex:

$$\cdots \xrightarrow{(\partial^X_{n+1})_*} (X_n)_* \xrightarrow{(\partial^X_n)_*} (X_{n-1})_* \xrightarrow{(\partial^X_{n-1})_*} \cdots$$

where $(-)_* = \operatorname{Hom}_R(M,-)$ and $(\partial^X_n)_*(f) = \partial^X_n f$.
(b) Prove that $\operatorname{Hom}_R(X,M)$ is isomorphic to the following complex:

$$\cdots \xrightarrow{(\partial^X_n)^*} X^*_n \xrightarrow{(\partial^X_{n+1})^*} X^*_{n+1} \xrightarrow{(\partial^X_{n+2})^*} \cdots .$$

where $(-)^* = \operatorname{Hom}_R(-,M)$ and $(\partial^X_n)^*(f) = f\partial^X_n$. [Hint: Mind the signs.]

Exercise 3.7 Let $f\colon X \to Y$ be a chain map.

(a) Prove that for each $i \in \mathbb{Z}$, the chain map f induces a well-defined R-module homomorphism $\mathrm{H}_i(f)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ given by $\mathrm{H}_i(f)(\overline{x}) := \overline{f_i(x)}$.
(b) Prove that if f is null-homotopic, then $\mathrm{H}_i(f) = 0$ for all $i \in \mathbb{Z}$.

The following concept is central for homological algebra; see, e.g., Remark 3.11.

Definition 3.8 A chain map $f\colon X \to Y$ is a *quasiisomorphism* if for all $i \in \mathbb{Z}$ the induced map $\mathrm{H}_i(f)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism. We use the symbol $\simeq$ to identify quasiisomorphisms.

Exercise 3.9 Prove that each isomorphism of R-complexes is a quasiisomorphism.

Exercise 3.10 Let M be an R-module with augmented projective resolution P^+ and augmented injective resolution ${}^+I$; see the notation from Example 3.2. Prove that τ and ε induce quasiisomorphisms $P \xrightarrow{\simeq} M \xrightarrow{\simeq} I$.

Remark 3.11 Let M and N be R-modules. The fact that $\mathrm{Ext}^i_R(M,N)$ can be computed using a projective resolution P of M or an injective resolution I of N is called the "balance" property for Ext. It can be proved by showing that there are quasiisomorphisms $\mathrm{Hom}_R(P,N) \xrightarrow{\simeq} \mathrm{Hom}_R(P,I) \xleftarrow{\simeq} \mathrm{Hom}_R(M,I)$. See Fact 3.15.

Hom and Chain Maps (Functoriality)

Given that the chain maps are the morphisms in the category of R-complexes, the next construction and the subsequent exercise indicate that $\mathrm{Hom}_R(Z,-)$ and $\mathrm{Hom}_R(-,Z)$ are functors.

Definition 3.12 Given a chain map $f\colon X \to Y$ and an R-complex Z, we define $\mathrm{Hom}_R(Z,f)\colon \mathrm{Hom}_R(Z,X) \to \mathrm{Hom}_R(Z,Y)$ as follows: each $\{g_p\} \in \mathrm{Hom}_R(Z,X)_n$ is mapped to $\{f_{p+n}g_p\} \in \mathrm{Hom}_R(Z,Y)_n$. Similarly, define $\mathrm{Hom}_R(f,Z)\colon \mathrm{Hom}_R(Y,Z) \to \mathrm{Hom}_R(X,Z)$ by the formula $\{g_p\} \mapsto \{g_p f_p\}$.

Remark 3.13 We do not use a sign-change in this definition because $|f| = 0$.

Exercise 3.14 Given a chain map $f\colon X \to Y$ and an R-complex Z, Prove that $\mathrm{Hom}_R(Z,f)$ and $\mathrm{Hom}_R(f,Z)$ are chain maps.

Fact 3.15 Let $f\colon X \xrightarrow{\simeq} Y$ be a quasiisomorphism, and let Z be an R-complex. In general, the chain map $\mathrm{Hom}_R(Z,f)\colon \mathrm{Hom}_R(Z,X) \to \mathrm{Hom}_R(Z,Y)$ is not a quasiisomorphism. However, if Z is a complex of projective R-modules such that $Z_i = 0$ for $i \ll 0$, then $Z \otimes_R f$ is a quasiisomorphism. Similarly, $\mathrm{Hom}_R(f,Z)\colon \mathrm{Hom}_R(Y,Z) \to \mathrm{Hom}_R(X,Z)$ is not a quasiisomorphism. However, if Z is a complex of injective R-modules such that $Z_i = 0$ for $i \gg 0$, then $\mathrm{Hom}_R(f,Z)$ is a quasiisomorphism.

Homotheties and Semidualizing Modules

We next explain how the Hom complex relates to the semidualizing modules from Section 2.

Exercise 3.16 Let X be an R-complex, and let $r \in R$. For each $p \in \mathbb{Z}$, let $\mu_p^{X,r}\colon X_p \to X_p$ be given by $x \mapsto rx$. (Such a map is a "homothety". When it is convenient, we denote this map as $X \xrightarrow{r} X$.)

Prove that $\mu^{X,r} := \{\mu_p^{X,r}\} \in \mathrm{Hom}_R(X,X)_0$ is a chain map. Prove that for all $i \in \mathbb{Z}$ the induced map $\mathrm{H}_i(\mu^{X,r})\colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is multiplication by r.

Exercise 3.17 Let X be an R-complex. We use the notation from Exercise 3.16. Define $\chi_0^X\colon R \to \mathrm{Hom}_R(X,X)$ by the formula $\chi_0^X(r) := \mu^{X,r} \in \mathrm{Hom}_R(X,X)_0$. Prove that this determines a chain map $\chi^X\colon R \to \mathrm{Hom}_R(X,X)$. The chain map χ^X is the "homothety morphism" for X.

Fact 3.18 Let M be a finitely generated R-module. We use the notation from Exercise 3.17. The following conditions are equivalent:

(i) M is a semidualizing R-module.
(ii) For each projective resolution P of M, the chain map $\chi^P\colon R \to \operatorname{Hom}_R(P,P)$ is a quasiisomorphism.
(iii) For some projective resolution P of M, the chain map $\chi^P\colon R \to \operatorname{Hom}_R(P,P)$ is a quasiisomorphism.
(iv) For each injective resolution I of M, the chain map $\chi^I\colon R \to \operatorname{Hom}_R(I,I)$ is a quasiisomorphism.
(v) For some injective resolution I of M, the chain map $\chi^I\colon R \to \operatorname{Hom}_R(I,I)$ is a quasiisomorphism.

In some sense, the point is that the homologies of the complexes $\operatorname{Hom}_R(P,P)$ and $\operatorname{Hom}_R(I,I)$ are exactly the modules $\operatorname{Ext}^i_R(M,M)$ by Fact 3.15.

4 Tensor Products and the Koszul Complex

Tensor products for complexes are as fundamental for complexes as they are for modules. In this section, we use them to construct the Koszul complex; see Definition 4.10. In Section 7, we use them for base change; see, e.g., Exercise 7.10.

Tensor Product of Complexes

As with the Hom complex, the sign convention in the next construction guarantees that it is an R-complex; see Exercise 4.2 and Remark 4.3. Note that Remark 4.4 describes a notational simplification.

Definition 4.1 Fix R-complexes X and Y. The *tensor product complex* $X \otimes_R Y$ is defined as follows. For each integer n, set $(X \otimes_R Y)_n := \bigoplus_{p\in\mathbb{Z}} X_p \otimes_R Y_{n-p}$ and let $\partial_n^{X\otimes_R Y}$ be given on generators by the formula $\partial_n^{X\otimes_R Y}(\ldots,0,x_p\otimes y_{n-p},0,\ldots) := (\ldots,0,\partial_p^X(x_p)\otimes y_{n-p},(-1)^p x_p\otimes\partial_{n-p}^Y(y_{n-p}),0,\ldots)$.

Exercise 4.2 Let X, Y, and Z be R-complexes.

(a) Prove that $X \otimes_R Y$ is an R-complex.
(b) Prove that there is a "tensor cancellation" isomorphism $R \otimes_R X \cong X$.
(c) Prove that there is a "commutativity" isomorphism $X \otimes_R Y \cong Y \otimes_R X$. (Hint: Mind the signs. This isomorphism is given by $x \otimes y \mapsto (-1)^{|x||y|} y \otimes x$.)
(d) Verify the "associativity" isomorphism $X \otimes_R (Y \otimes_R Z) \cong (X \otimes_R Y) \otimes_R Z$.

Remark 4.3 There is a rule of thumb for sign conventions like the one in the hint for Exercise 4.2: whenever two factors u and v are commuted in an expression, you multiply by $(-1)^{|u||v|}$. This can already be seen in $\partial^{\mathrm{Hom}_R(X,Y)}$ and $\partial^{X\otimes_R Y}$. This graded commutativity is one of the keys to DG algebra. See Section 5.

Remark 4.4 After working with the tensor product of complexes for a few moments, one realizes that the sequence notation $(\ldots,0,x_p\otimes y_{n-p},0,\ldots)$ is unnecessarily cumbersome. We use the sequence notation in a few of the solutions in Appendix 10, but not for many of them. Similarly, from now on, instead of writing $(\ldots,0,x_p\otimes y_{n-p},0,\ldots)$, we write the simpler $x_p\otimes y_{n-p}$. As we note in 10.11, one needs to be somewhat careful with this notation, as elements $u\otimes v$ and $x\otimes y$ only live in the same summand when $|u|=|x|$ and $|v|=|y|$.

Fact 4.5 Given R-complexes $X^1,\ldots,X^n$ an induction argument using the associativity isomorphism from Exercise 4.2 shows that the n-fold tensor product $X^1\otimes_R\cdots\otimes_R X^n$ is well-defined (up to isomorphism).

Tensor Products and Chain Maps (Functoriality)

As for Hom, the next items indicate that $Z\otimes_R -$ and $-\otimes_R Z$ are functors.

Definition 4.6 Consider a chain map $f\colon X\to Y$ and an R-complex Z. We define $Z\otimes_R f\colon Z\otimes_R X\to Z\otimes_R Y$ by the formula $z\otimes y\mapsto z\otimes f(y)$. Similarly, define the map $f\otimes_R Z\colon X\otimes_R Z\to Y\otimes_R Z$ by the formula $x\otimes z\mapsto f(x)\otimes z$.

Remark 4.7 We do not use a sign-change in this definition because $|f|=0$.

Exercise 4.8 Consider a chain map $f\colon X\to Y$ and an R-complex Z. Then the maps $Z\otimes_R f\colon Z\otimes_R X\to Z\otimes_R Y$ and $f\otimes_R Z\colon X\otimes_R Z\to Y\otimes_R Z$ are chain maps.

Fact 4.9 Let $f\colon X\xrightarrow{\simeq} Y$ be a quasiisomorphism, and let Z be an R-complex. In general, the chain map $Z\otimes_R f\colon Z\otimes_R X\to Z\otimes_R Y$ is not a quasiisomorphism. However, if Z is a complex of projective R-modules such that $Z_i=0$ for $i\ll 0$, then $Z\otimes_R f$ is a quasiisomorphism.

The Koszul Complex

Here begins our discussion of the prototypical DG algebra.

Definition 4.10 Let $\mathbf{x}=x_1,\ldots,x_n\in R$. For $i=1,\ldots,n$ set

$$K^R(x_i)=\quad 0\to R\xrightarrow{x_i}R\to 0.$$

Using Remark 4.5, we set

$$K^R(\mathbf{x})=K^R(x_1,\ldots,x_n)=K^R(x_1)\otimes_R\cdots\otimes_R K^R(x_n).$$

Exercise 4.11 Let $x,y,z \in R$. Write out explicit formulas, using matrices for the differentials, for $K^R(x,y)$ and $K^R(x,y,z)$.

Exercise 4.12 Let $\mathbf{x} = x_1,\ldots,x_n \in R$. Prove that $K^R(\mathbf{x})_i \cong R^{\binom{n}{i}}$ for all $i \in \mathbb{Z}$. (Here we use the convention $\binom{n}{i} = 0$ for all $i < 0$ and $i > n$.)

Exercise 4.13 Let $\mathbf{x} = x_1,\ldots,x_n \in R$. Let $\sigma \in S_n$, and set $\mathbf{x}' = x_{\sigma(1)},\ldots,x_{\sigma(n)}$. Prove that $K^R(\mathbf{x}) \cong K^R(\mathbf{x}')$.

Given a generating sequence $\mathbf{x}$ for the maximal ideal of a local ring R, one concludes from the next lemma that each homology module $\mathrm{H}_i(K^R(\mathbf{x}))$ has finite length. This is crucial for the proof of Theorem 1.4.

Lemma 4.14 *If $\mathbf{x} = x_1,\ldots,x_n \in R$ and $\mathfrak{a} = (\mathbf{x})R$, then $\mathfrak{a}\,\mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$.*

Proof (Sketch of proof). It suffices to show that for $j = 1,\ldots,n$ and for all $i \in \mathbb{Z}$ we have $x_j\,\mathrm{H}_i(K^R(\mathbf{x})) = 0$. By symmetry (Exercise 4.13) it suffices to show that $x_1\,\mathrm{H}_i(K^R(\mathbf{x})) = 0$ for all $i \in \mathbb{Z}$ for $j = 1,\ldots,n$. The following diagram shows that the chain map $K^R(x_1) \xrightarrow{x_1} K^R(x_1)$ is null-homotopic.

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "x_1"] \arrow[d, "x_1"] \arrow[dl] & R \arrow[r] \arrow[d, "x_1"] \arrow[dl, "1"] & 0 \arrow[dl] \\
0 \arrow[r] & R \arrow[r, "x_1"'] & R \arrow[r] & 0
\end{tikzcd}$$

It is routine to show that this implies that the induced map $K^R(\mathbf{x}) \xrightarrow{x_1} K^R(\mathbf{x})$ is null-homotopic. The desired conclusion now follows from Exercise 3.7. $\square$

The next construction allows us to push our complexes around.

Definition 4.15 Let X be an R-complex, and let $n \in \mathbb{Z}$. The *nth suspension* (or *shift*) of X is the complex $\Sigma^n X$ such that $(\Sigma^n X)_i := X_{i-n}$ and $\partial_i^{\Sigma^n X} = (-1)^n \partial_{i-n}^X$. We set $\Sigma X := \Sigma^1 X$.

The next fact is in general quite useful, though we do not exploit it here.

Fact 4.16 Let $\mathbf{x} = x_1,\ldots,x_n \in R$. The Koszul complex $K^R(\mathbf{x})_i$ is "self-dual", that is, that there is an isomorphism of R-complexes $\mathrm{Hom}_R(K^R(\mathbf{x}),R) \cong \Sigma^n K^R(\mathbf{x})$. (This fact is not trivial.)

Exercise 4.17 Verify the isomorphism from Fact 4.16 for $n = 1,2,3$.

The following result gives the first indication of the utility of the Koszul complex. We use it explicitly in the proof of Theorem 1.4.

Lemma 4.18 *Let $\mathbf{x} = x_1,\ldots,x_n \in R$. If $\mathbf{x}$ is R-regular, then $K^R(\mathbf{x})$ is a free resolution of $R/(\mathbf{x})$ over R.*

Proof. Argue by induction on n.

Base case: $n=1$. Assume that x_1 is R-regular. Since $K^R(x_1)$ has the form $0 \to R \xrightarrow{x_1} R \to 0$, the fact that x_1 is R-regular implies that $\mathrm{H}_i(K^R(x_1)) = 0$ for all $i \neq 0$. As each module in $K^R(x_1)$ is free, it follows that $K^R(x_1)$ is a free resolution of $R/(x_1)$.

Inductive step: Assume that $n \geqslant 2$ and that the result holds for regular sequences of length $n-1$. Assume that $\mathbf{x}$ is R-regular (of length n). Thus, the sequence $\mathbf{x}' = x_1, \ldots, x_{n-1}$ is R-regular, and x_n is $R/(\mathbf{x}')$-regular. The first condition implies that $K' := K^R(\mathbf{x}')$ is a free resolution of $R/(\mathbf{x}')$ over R. By definition, we have $K := K^R(\mathbf{x}) = K' \otimes_R K^R(x_n)$. Further, by definition of $K' \otimes_R K^R(x_n)$, we have

$$K \cong \cdots \xrightarrow{\begin{pmatrix} \partial^{K'}_{i+1} & (-1)^i x_n \\ 0 & \partial^{K'}_i \end{pmatrix}} \begin{matrix} K'_i \\ \bigoplus \\ K'_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial^{K'}_{i} & (-1)^{i-1} x_n \\ 0 & \partial^{K'}_{i-1} \end{pmatrix}} \begin{matrix} K'_{i-1} \\ \bigoplus \\ K'_{i-2} \end{matrix} \xrightarrow{\begin{pmatrix} \partial^{K'}_{i-1} & (-1)^{i-2} x_n \\ 0 & \partial^{K'}_{i-2} \end{pmatrix}} \cdots.$$

(Note that the term $K'_i \bigoplus K'_{i-1}$ is shorthand for $(K'_i \otimes_R R1) \bigoplus (K'_{i-1} \otimes_R Re)$.) Using this, there is a short exact sequence of R-complexes and chain maps[3]

$$0 \longrightarrow K' \longrightarrow K \longrightarrow K'' \longrightarrow 0$$

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial^{K'}_{i+1}"'] & \vdots \arrow[d, "\partial^{K}_{i+1}"'] & \vdots \arrow[d, "\partial^{K'}_{i}"'] & \\
0 \arrow[r] & K'_i \arrow[r] \arrow[d, "\partial^{K'}_{i}"'] & K_i \arrow[r] \arrow[d, "\partial^{K}_{i}"'] & K'_{i-1} \arrow[r] \arrow[d, "\partial^{K'}_{i-1}"'] & 0 \\
0 \arrow[r] & K'_{i-1} \arrow[r] \arrow[d, "\partial^{K'}_{i-1}"'] & K_{i-1} \arrow[r] \arrow[d, "\partial^{K}_{i-1}"'] & K'_{i-2} \arrow[r] \arrow[d, "\partial^{K'}_{i-2}"'] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where K'' is obtained by shifting K'.[4] Furthermore, it can be shown that the long exact sequence in homology has the form

$$\cdots \mathrm{H}_i(K') \xrightarrow{(-1)^i x_n} \mathrm{H}_i(K') \to \mathrm{H}_i(K) \to \mathrm{H}_i(K') \xrightarrow{(-1)^{i-1} x_n} \mathrm{H}_i(K') \to \cdots.$$

Since K' is a free resolution of $R/(\mathbf{x}')$ over R, we have $\mathrm{H}_i(K') = 0$ for all $i \neq 0$, and $\mathrm{H}_0(K') \cong R/(\mathbf{x}')$. As x_n is $R/(\mathbf{x}')$-regular, an analysis of the long exact sequence shows that $\mathrm{H}_i(K) = 0$ for all $i \neq 0$, and $\mathrm{H}_0(K) \cong R/(\mathbf{x})$. It follows that K is a free resolution of $R/(\mathbf{x})$, as desired. $\square$

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[3] Readers who know the mapping cone description of K should not be surprised by this argument.

[4] Note that K'' is technically not equal to the complex $\Sigma K'$ from Definition 4.15, since there is no sign on the differential. On the other hand the complexes K'' and $\Sigma K'$ are isomorphic.

Alternate Description of the Koszul Complex

The following description of $K^R(\mathbf{x})$ says that $K^R(\mathbf{x})$ is given by the "exterior algebra" on R^n; see Fact 4.22.

Definition 4.19 Let $\mathbf{x} = x_1,\dots,x_n \in R$. Fix a basis $e_1,\dots,e_n \in R^n$. For $i > 1$, set $\bigwedge^i R^n := R^{\binom{n}{i}}$ with basis given by the set of formal symbols $e_{j_1} \wedge \cdots \wedge e_{j_i}$ such that $1 \leqslant j_1 < \cdots < j_i \leqslant n$. This extends to all $i \in \mathbb{Z}$ as follows: $\bigwedge^1 R^n = R^n$ with basis $e_1,\dots,e_n$ and $\bigwedge^0 R^n = R^1$ with basis 1; for $i < 0$, set $\bigwedge^i R^n = R^{\binom{n}{i}} = 0$.

Define $\widetilde{K}^R(\mathbf{x})$ as follows. For all $i \in \mathbb{Z}$ set $\widetilde{K}^R(\mathbf{x})_i = \bigwedge^i R^n$, and let $\partial_i^{\widetilde{K}^R(\mathbf{x})}$ be given on basis vectors by the formula

$$\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{p=1}^{i} (-1)^{p-1} x_{j_p} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_p}} \wedge \cdots \wedge e_{j_i}$$

where the notation $\widehat{e_{j_p}}$ indicates that e_{j_p} has been removed from the list. In the case $i = 1$, the formula reads as $\partial_1^{\widetilde{K}^R(\mathbf{x})}(e_j) = x_j$.

Remark 4.20 Our definition of $\bigwedge^i R^n$ is *ad hoc*. A better way to think about it (in some respects) is in terms of a universal mapping property for alternating multilinear maps. A basis-free construction can be given in terms of a certain quotient of the i-fold tensor product $R^n \otimes_R \cdots \otimes_R R^n$.

Exercise 4.21 Let $\mathbf{x} = x_1,\dots,x_n \in R$. Write out explicit formulas, using matrices for the differentials, for $\widetilde{K}^R(\mathbf{x})$ in the cases $n = 1,2,3$.

Fact 4.22 Let $\mathbf{x} = x_1,\dots,x_n \in R$. There is an isomorphism of R-complexes $K^R(\mathbf{x}) \cong \widetilde{K}^R(\mathbf{x})$. (This fact is not trivial. For perspective on this, compare the solutions to Exercises 4.11 and 4.21 in 10.13 and 10.17.) In light of this fact, we do not distinguish between $K^R(\mathbf{x})$ and $\widetilde{K}^R(\mathbf{x})$ for the remainder of these notes.

Remark 4.23 A third description of $K^R(\mathbf{x})$ involves the mapping cone. Even though it is extremely useful, we do not discuss it in detail here.

Algebra Structure on the Koszul Complex

In our estimation, the Koszul complex is one of the most important constructions in commutative algebra. When the sequence $\mathbf{x}$ is R-regular, it is an R-free resolution of $R/(\mathbf{x})$, by Lemma 4.18. In general, it detects depth and has all scads of other magical properties. For us, one its most important features is its algebra structure, which we describe next.

Definition 4.24 Let $n \in \mathbb{N}$ and let $e_1,\dots,e_n \in R^n$ be a basis. In $\bigwedge^2 R^n$, define

$$e_{j_2} \wedge e_{j_1} := \begin{cases} -e_{j_1} \wedge e_{j_2} & \text{whenever } 1 \leqslant j_1 < j_2 \leqslant n \\ 0 & \text{whenever } 1 \leqslant j_1 = j_2 \leqslant n. \end{cases}$$

Extending this bilinearly, we define $\alpha \wedge \beta$ for all $\alpha, \beta \in \bigwedge^1 R^n = R^n$: write $\alpha = \sum_p \alpha_p e_p$ and $\beta = \sum_q \beta_q e_q$, and define

$$\alpha \wedge \beta = \left(\sum_p \alpha_p e_p\right) \wedge \left(\sum_q \beta_q e_q\right) = \sum_{p,q} \alpha_p \beta_q e_p \wedge e_q = \sum_{p<q} (\alpha_p \beta_q - \alpha_q \beta_p) e_p \wedge e_q.$$

This extends to a multiplication $\bigwedge^1 R^n \times \bigwedge^t R^n \to \bigwedge^{1+t} R^n$ using the following formula, assuming that $1 \leqslant i \leqslant n$ and $1 \leqslant j_1 < \cdots < j_t \leqslant n$:

$$e_i \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) := \begin{cases} 0 & \text{if } i = j_p \text{ for some } p \\ e_i \wedge e_{j_1} \wedge \cdots \wedge e_{j_t} & \text{if } i < j_1 \\ (-1)^p e_{j_1} \wedge \cdots \wedge e_{j_p} \wedge e_i \wedge \cdots \wedge e_{j_t} & \text{if } j_p < i < j_{p+1} \\ (-1)^t e_{j_1} \wedge \cdots \wedge e_{j_t} \wedge e_i & \text{if } j_t < i. \end{cases}$$

This extends (by induction on s) to a multiplication $\bigwedge^s R^n \times \bigwedge^t R^n \to \bigwedge^{s+t} R^n$ using the following formula when $i_1 < \ldots < i_s$ and $1 \leqslant j_1 < \cdots < j_t \leqslant n$:

$$(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) := e_{i_1} \wedge [(e_{i_2} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t})].$$

This multiplication is denoted as $(\alpha, \beta) \mapsto \alpha \wedge \beta$. When $s = 0$, since $\bigwedge^0 R^n = R$, the usual scalar multiplication $R \times \bigwedge^t R^n \to \bigwedge^t R^n$ describes the rule for multiplication $\bigwedge^0 R^n \times \bigwedge^t R^n \to \bigwedge^t R^n$, and similarly when $t = 0$. This further extends to a well-defined multiplication on $\bigwedge R^n := \bigoplus_i \bigwedge^i R^n$.

Remark 4.25 According to Definition 4.24, for $0 \in \bigwedge^s R^n$ and $\beta \in \bigwedge^t R^n$, we have $0 \wedge \beta = 0 = \beta \wedge 0$.

Example 4.26 We compute a few products in $\bigwedge R^4$:

$$\begin{aligned} (e_1 \wedge e_2) \wedge (e_3 \wedge e_4) &= e_1 \wedge e_2 \wedge e_3 \wedge e_4 \\ (e_1 \wedge e_2) \wedge (e_2 \wedge e_3) &= 0 \\ (e_1 \wedge e_3) \wedge (e_2 \wedge e_4) &= e_1 \wedge [e_3 \wedge (e_2 \wedge e_4)] \\ &= e_1 \wedge [-e_2 \wedge e_3 \wedge e_4] \\ &= -e_1 \wedge e_2 \wedge e_3 \wedge e_4. \end{aligned}$$

Exercise 4.27 Write out multiplication tables (for basis vectors only) for $\bigwedge R^n$ with $n = 1, 2, 3$.

Definition 4.24 suggests the next notation, which facilitates many computations. The subsequent exercises exemplify this, culminating in the important Exercise 4.33

Definition 4.28 Let $n \in \mathbb{N}$, let $j_1,\dots,j_t \in \{1,\dots,n\}$, and let $e_1,\dots,e_n \in R^n$ be a basis. Since multiplication of basis vectors in $\bigwedge R^n$ is defined inductively, the following element (also defined inductively) is well-defined.

$$e_{j_1} \wedge \cdots \wedge e_{j_t} := \begin{cases} 0 & \text{if } j_p = j_q \text{ for some } p \neq q \\ e_{j_1} \wedge (e_{j_2} \wedge \cdots \wedge e_{j_t}) & \text{if } j_p \neq j_q \text{ for all } p \neq q \end{cases}$$

Exercise 4.29 Let $n \in \mathbb{N}$, and let $e_1,\dots,e_n \in R^n$ be a basis. Let $j_1,\dots,j_t \in \{1,\dots,n\}$ such that $j_p \neq j_q$ for all $p \neq q$, and let $\iota \in S_n$ such that ι fixes all elements of $\{1,\dots,n\} \smallsetminus \{j_1,\dots,j_t\}$. Prove that

$$e_{\iota(j_1)} \wedge \cdots \wedge e_{\iota(j_t)} = \operatorname{sgn}(\iota) e_{j_1} \wedge \cdots \wedge e_{j_t}$$

Exercise 4.30 Let $n \in \mathbb{N}$, and let $e_1,\dots,e_n \in R^n$ be a basis. Prove that for all $i_1,\dots,i_s,j_1,\dots,j_t \in \{1,\dots,n\}$ we have

$$(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) = e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}$$

and that this is 0 if there is a repetition in the list $i_1,\dots,i_s,j_1,\dots,j_t$. (The points here are the order on the subscripts matter in Definition 4.24 and do not matter in Definition 4.28, so one needs to make sure that the signs that occur from Definition 4.24 agree with those from Definition 4.28.)

Exercise 4.31 Let $n \in \mathbb{N}$ and let $e_1,\dots,e_n \in R^n$ be a basis. Prove that the multiplication from Definition 4.24 makes $\bigwedge R^n$ into a graded commutative R-algebra:

(a) multiplication in $\bigwedge R^n$ is associative, distributive, and unital;
(b) for elements $\alpha \in \bigwedge^s R^n$ and $\beta \in \bigwedge^t R^n$, we have $\alpha \wedge \beta = (-1)^{st} \beta \wedge \alpha$;
(c) for $\alpha \in \bigwedge^s R^n$, if s is odd, then $\alpha \wedge \alpha = 0$; and
(d) the composition $R \xrightarrow{\cong} \bigwedge^0 R^n \xrightarrow{\subseteq} \bigwedge R^n$ is a ring homomorphism, the image of which is contained in the center of $\bigwedge R^n$.

Hint: The distributive law holds essentially by definition. For the other properties in (a) and (b), prove the desired formula for basis vectors, then verify it for general elements using distributivity.

Exercise 4.32 Let $n \in \mathbb{N}$ and let $e_1,\dots,e_n \in R^n$ be a basis. Let $\mathbf{x} = x_1,\dots,x_n \in R$, and let $j_1,\dots,j_t \in \{1,\dots,n\}$. Prove that the element $e_{j_1} \wedge \cdots \wedge e_{j_t}$ from Definition 4.28 satisfies the following formula:

$$\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1} \wedge \cdots \wedge e_{j_t}) = \sum_{s=1}^{t} (-1)^{s+1} x_{j_s} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_s}} \wedge \cdots \wedge e_{j_t}.$$

(Note that, if $j_1 < \cdots < j_t$, then this is the definition of $\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1} \wedge \cdots \wedge e_{j_t})$. However, we are not assuming that $j_1 < \cdots < j_t$.)

Exercise 4.33 Let $n \in \mathbb{N}$ and let $e_1, \dots, e_n \in R^n$ be a basis. Let $\mathbf{x} = x_1, \dots, x_n \in R$. Prove that the multiplication from Definition 4.24 satisfies the "Leibniz rule": for elements $\alpha \in \bigwedge^s R^n$ and $\beta \in \bigwedge^t R^n$, we have

$$\partial_{s+t}^{\widetilde{K}^R(\mathbf{x})}(\alpha \wedge \beta) = \partial_s^{\widetilde{K}^R(\mathbf{x})}(\alpha) \wedge \beta + (-1)^s \alpha \wedge \partial_t^{\widetilde{K}^R(\mathbf{x})}(\beta).$$

Hint: Prove the formula for basis vectors and verify it for general elements using distributivity and linearity.

5 DG Algebras and DG Modules I

This section introduces the main tools for the proof of Theorem 1.4. This proof is outlined in 5.32. From the point of view of this proof, the first important example of a DG algebra is the Koszul complex; see Example 5.3. However, the proof showcases another important example, namely, the DG algebra resolution of Definition 5.29.

DG Algebras

The first change of perspective required for the proof of Theorem 1.4 is the change from rings to DG algebras.

Definition 5.1 A *commutative differential graded algebra over R* (*DG R-algebra* for short) is an R-complex A equipped with a binary operation $A \times A \to A$, written as $(a,b) \mapsto ab$ and called the *product* on A, satisfying the following properties:[5]

- associative: for all $a,b,c \in A$ we have $(ab)c = a(bc)$;
- distributive: for all $a,b,c \in A$ such that $|a| = |b|$ we have $(a+b)c = ac+bc$ and $c(a+b) = ca+cb$;
- unital: there is an element $1_A \in A_0$ such that for all $a \in A$ we have $1_A a = a$;
- graded commutative: for all $a,b \in A$ we have $ba = (-1)^{|a||b|}ab \in A_{|a|+|b|}$, and $a^2 = 0$ when $|a|$ is odd;
- positively graded: $A_i = 0$ for $i < 0$; and
- Leibniz Rule: for all $a,b \in A$ we have

$$\partial^A_{|a|+|b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|} a \partial^A_{|b|}(b).$$

[5] We assume that readers at this level are familiar with associative laws and the like. However, given that the DG universe is riddled with sign conventions, we explicitly state these laws for the sake of clarity.

Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \bigoplus_{i=0}^{\infty} A_i$. When R is a field and $\operatorname{rank}_R(\bigoplus_{i\geqslant 0} A_i) < \infty$, we say that A is *finite-dimensional* over R.

It should be helpful for the reader to keep the next two examples in mind for the remainder of these notes.

Example 5.2 The ring R, considered as a complex concentrated in degree 0, is a DG R-algebra such that $R^\natural = R$.

Example 5.3 Given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, the Koszul complex $K = K^R(\mathbf{x})$ is a DG R-algebra such that $K^\natural = \bigwedge R^n$; see Exercises 4.31 and 4.33. In particular, if $n = 1$, then $K^\natural \cong R[X]/(X^2)$ where $|X| = 1$.

The following exercise is a routine interpretation of the Leibniz Rule. It is also an important foreshadowing of the final part of the proof of Theorem 1.4 which is given in Section 8. See also Exercise 5.15.

Exercise 5.4 Prove that if A is a DG R-algebra, then there is a well-defined chain map $\mu^A\colon A \otimes_R A \to A$ given by $\mu^A(a \otimes b) = ab$, and that A_0 is an R-algebra. (Interested readers may wish to formulate and prove the converse to this statement also.)

The next notion will allow us to transfer information from one DG algebra to another as in the arguments for $R \to \widehat{R}$ and $R \to \overline{R}$ described in Section 1.

Definition 5.5 A *morphism* of DG R-algebras is a chain map $f\colon A \to B$ between DG R-algebras respecting products and multiplicative identities: $f(aa') = f(a)f(a')$ and $f(1_A) = 1_B$. A morphism of DG R-algebras that is also a quasiisomorphism is a *quasiisomorphism of DG R-algebras*.

Part (a) of the next exercise contains the first morphism of DG algebras that we use in the proof of Theorem 1.4.

Exercise 5.6 Let A be a DG R-algebra.

(a) Prove that the map $R \to A$ given by $r \mapsto r \cdot 1_A$ is a morphism of DG R-algebras. As a special case, given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, the natural map $R \to K = K^R(\mathbf{x})$ given by $r \mapsto r \cdot 1_K$ is a morphism of DG R-algebras.
(b) Prove that the natural inclusion map $A_0 \to A$ is a morphism of DG R-algebras.
(c) Prove that the natural map $K \to R/(\mathbf{x})$ is a morphism of DG R-algebras that is a quasiisomorphism if $\mathbf{x}$ is R-regular; see Exercise 3.10 and Lemma 4.18.

Part (b) of the next exercise is needed for the subsequent definition.

Exercise 5.7 Let A be a DG R-algebra.

(a) Prove that the condition $A_{-1} = 0$ implies that A_0 surjects onto $\mathrm{H}_0(A)$ and that $\mathrm{H}_0(A)$ is an A_0-algebra.
(b) Prove that A_i is an A_0-module, and $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module for each i.

Remark 5.8 Let A be a DG R-algebra. Then the subset $Z(A) := \bigoplus_{i=0}^{\infty} \operatorname{Ker}(\partial_i^A)$ is a graded R-algebra. Also, the submodule $B(A) := \bigoplus_{i=0}^{\infty} \operatorname{Im}(\partial_{i+1}^A)$ is a graded ideal of $Z(A)$, so the quotient $\mathrm{H}(A) = B(A)/Z(A) = \bigoplus_{i=0}^{\infty} \mathrm{H}_i(A)$ is a graded R-algebra.

Definition 5.9 Let A be a DG R-algebra. We say that A is *noetherian* if $\mathrm{H}_0(A)$ is noetherian and $\mathrm{H}_i(A)$ is finitely generated over $\mathrm{H}_0(A)$ for all $i \geqslant 0$.

Exercise 5.10 Given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, prove that the Koszul complex $K^R(\mathbf{x})$ is a noetherian DG R-algebra. Moreover, prove that any DG R-algebra A such that each A_i is finitely generated over R is noetherian.

DG Modules

In the passage from rings to DG algebras, modules and complexes change to DG modules, which we describe next.

Definition 5.11 Let A be a DG R-algebra, and let i be an integer. A *differential graded module over A* (*DG A-module* for short) is an R-complex M equipped with a binary operation $A \times M \to M$, written as $(a,m) \mapsto am$ and called the *scalar multiplication* of A on M, satisfying the following properties:

- associative: for all $a, b \in A$ and $m \in M$ we have $(ab)m = a(bm)$;
- distributive: for all $a, b \in A$ and $m, n \in M$ such that $|a| = |b|$ and $|m| = |n|$, we have $(a+b)m = am + bm$ and $a(m+n) = am + an$;
- unital: for all $m \in M$ we have $1_A m = m$;
- graded: for all $a \in A$ and $m \in M$ we have $am \in M_{|a|+|m|}$;
- Leibniz Rule: for all $a \in A$ and $m \in M$ we have

$$\partial^A_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|} a \partial^M_{|m|}(m).$$

The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \bigoplus_{j=-\infty}^{\infty} M_j$.

The ith *suspension* of a DG A-module M is the DG A-module $\Sigma^i M$ defined by $(\Sigma^i M)_n := M_{n-i}$ and $\partial_n^{\Sigma^i M} := (-1)^i \partial^M_{n-i}$. The scalar multiplication on $\Sigma^i M$ is defined by the formula $a * m := (-1)^{i|a|} am$. The notation ΣM is short for $\Sigma^1 M$.

The next exercise contains examples that should be helpful to keep in mind.

Exercise 5.12

(a) Prove that DG R-module is just an R-complex.
(b) Given a DG R-algebra A, prove that the complex A is a DG A-module where the scalar multiplication is just the internal multiplication on A.
(c) Given a morphism $A \to B$ of DG R-algebras, prove that every DG B-module is a DG A-module by restriction of scalars. As a special case, given a sequence $\mathbf{x} = x_1, \cdots, x_n \in R$, every $R/(\mathbf{x})$-complex is a DG $K^R(\mathbf{x})$-module; see Exercise 5.6.

The operation $X \mapsto A \otimes_R X$ described next is "base change", which is crucial for our passage between DG algebras in the proof of Theorem 1.4.

Exercise 5.13 Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$. Given an R-module M, prove that the complex $K \otimes_R M$ is a DG K-module via the multiplication $a(b \otimes m) := (ab) \otimes m$. More generally, given an R-complex X and a DG R-algebra A, prove that the complex $A \otimes_R X$ is a DG A-module via the multiplication $a(b \otimes x) := (ab) \otimes x$.

Exercise 5.14 Let A be a DG R-algebra, let M be a DG A-module, and let $i \in \mathbb{Z}$. Prove that $\Sigma^i M$ is a DG A-module.

The next exercise further foreshadows important aspects of Section 8.

Exercise 5.15 Let A be a DG R-algebra, and let M be a DG A-module. Prove that there is a well-defined chain map $\mu^M : A \otimes_R M \to M$ given by $\mu^M(a \otimes m) = am$.

We consider the following example throughout these notes. It is simple but demonstrates our constructions. And even it has some non-trivial surprises.

Example 5.16 We consider the trivial Koszul complex $U = K^R(0)$:

$$U = \quad 0 \to Re \xrightarrow{0} R1 \to 0.$$

The notation indicates that we are using the basis $e \in U_1$ and $1 = 1_U \in U_0$.

Exercise 5.13 shows that R is a DG U-module. Another example is the following, again with specified basis in each degree:

$$G = \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0.$$

The notation for the bases is chosen to help remember the DG U-module structure:

$$\begin{aligned} 1 \cdot 1_{2n} &= 1_{2n} & \qquad 1 \cdot e_{2n+1} &= e_{2n+1} \\ e \cdot 1_{2n} &= e_{2n+1} & \qquad e \cdot e_{2n+1} &= 0. \end{aligned}$$

One checks directly that G satisfies the axioms to be a DG U-module. It is worth noting that $\mathrm{H}_0(G) \cong R$ and $\mathrm{H}_i(G) = 0$ for all $i \neq 0$.[6]

We continue with Example 5.16, but working over a field F instead of R.

Example 5.17 We consider the trivial Koszul complex $U = K^F(0)$:

$$U = \quad 0 \to Fe \xrightarrow{0} F1 \to 0.$$

Consider the graded vector space $W = \bigoplus_{i \in \mathbb{Z}} W_i$, where $W_0 = F\eta_0 \cong F$ with basis η_0 and $W_i = 0$ for $i \neq 0$:

$$W = \quad 0 \bigoplus F\nu_0 \bigoplus 0.$$

[6] For perspective, G is modeled on the free resolution $\cdots \xrightarrow{e} R[e]/(e^2) \xrightarrow{e} R[e]/(e^2) \to 0$ of R over the ring $R[e]/(e^2)$.

We are interested in identifying all the possible DG U-module structures on W, that is, all possible differentials

$$0 \to F v_0 \to 0$$

and rules for scalar multiplication making this into a DG U-module. See Section 8 for more about this.

The given vector space W has exactly one DG U-module structure. To see this, first note that we have no choice for the differential since it maps $W_i \to W_{i-1}$ and at least one of these modules is 0; hence $\partial_i = 0$ for all i. Also, we have no choice for the scalar multiplication: multiplication by $1 = 1_U$ must be the identity, and multiplication by e maps $W_i \to W_{i+1}$ and at least one of these modules is 0. (This example is trivial, but it will be helpful later.)

Similarly, we consider the graded vector space

$$W' = \quad 0 \bigoplus F\eta_1 \bigoplus F\eta_0 \bigoplus 0.$$

This vector space allows for one possibly non-trivial differential

$$\partial_1' \in \mathrm{Hom}_F(F\eta_1, F\eta_0) \cong F.$$

So, in order to make W' into an R-complex, we need to choose an element $x_1 \in F$:

$$(W', x_1) = \quad 0 \to F\eta_1 \xrightarrow{x_1} F\eta_0 \to 0.$$

To be explicit, this means that $\partial_1'(\eta_1) = x_1\eta_0$, and hence $\partial_1'(r\eta_1) = x_1 r\eta_0$ for all $r \in F$. Since W' is concentrated in degrees 0 and 1, this is an R-complex.

For the scalar multiplication of U on the complex (W', x_1), again multiplication by 1 must be the identity, but multiplication by e has one nontrivial option

$$\mu_0' \in \mathrm{Hom}_F(F\eta_0, F\eta_1) \cong F$$

which we identify with an element $x_0 \in F$. To be explicit, this means that $e\eta_0 = x_0\eta_1$, and hence $er\eta_0 = x_0 r\eta_1$ for all $r \in F$.

For the Leibniz Rule to be satisfied, we must have

$$\partial_{i+1}'(e \cdot \eta_i) = \partial_1^U(e) \cdot \eta_i + (-1)^{|e|} e \cdot \partial_i'(\eta_i)$$

for $i = 0, 1$. We begin with $i = 0$:

$$\begin{aligned}
\partial_1'(e \cdot \eta_0) &= \partial_1^U(e) \cdot \eta_0 + (-1)^{|e|} e \cdot \partial_0'(\eta_0) \\
\partial_1'(x_0\eta_1) &= 0 \cdot \eta_0 - e \cdot 0 \\
x_0\partial_1'(\eta_1) &= 0 \\
x_0 x_1 \eta_0 &= 0
\end{aligned}$$

so we have $x_0 x_1 = 0$, that is, either $x_0 = 0$ or $x_1 = 0$. For $i = 1$, we have

$$
\begin{aligned}
\partial_2'(e\cdot\eta_1) &= \partial_1^U(e)\cdot\eta_1 + (-1)^{|e|}e\cdot\partial_1'(\eta_1)\\
0 &= 0\cdot\eta_1 - e\cdot(x_1\eta_0)\\
0 &= -x_1 e\cdot\eta_0\\
0 &= -x_1x_0\eta_1
\end{aligned}
$$

so we again conclude that $x_0 = 0$ or $x_1 = 0$. One can check the axioms from Definition 5.11 to see that either of these choices gives rise to a DG U-module structure on W'. In other words, the DG U-module structures on W' are parameterized by the following algebraic subset of $F^2 = \mathbb{A}^2_F$

$$\{(x_0,x_1)\in\mathbb{A}^2_F \mid x_0x_1 = 0\} = V(x_0)\cup V(x_1)$$

which is the union of the two coordinate axes in $\mathbb{A}^2_F$. This is one of the fundamental points of Section 8, that DG module structures on a fixed finite-dimensional graded vector space are parametrized by algebraic varieties.

Homologically finite DG modules, defined next, take the place of finitely generated modules in our passage to the DG universe.

Definition 5.18 Let A be a DG R-algebra. A DG A-module M is *bounded below* if $M_n = 0$ for all $n \ll 0$; and it is *homologically finite* if each $\mathrm{H}_0(A)$-module $\mathrm{H}_n(M)$ is finitely generated and $\mathrm{H}_n(M) = 0$ for $|n| \gg 0$.

Example 5.19 In Exercise 5.13, the DG K-module $R/(\mathbf{x})$ is bounded below and homologically finite. In Example 5.16, the DG U-modules R and G are bounded below and homologically finite. In Example 5.17, the DG U-module structures on W and W' are bounded below and homologically finite.

Morphisms of DG Modules

In the passage from modules and complexes to DG modules, homomorphisms and chain maps are replaced with morphisms.

Definition 5.20 A *morphism* of DG A-modules is a chain map $f\colon M\to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. A *quasiisomorphism* of DG A-modules is a morphism $M\to N$ such that each induced map $\mathrm{H}_i(M)\to\mathrm{H}_i(N)$ is an isomorphism, i.e., a morphism of DG A-modules that is a quasiisomorphism of R-complexes; these are identified by the symbol $\simeq$.

Remark 5.21 A morphism of DG R-modules is simply a chain map, and a quasiisomorphism of DG R-modules is simply a quasiisomorphism in the sense of Definition 3.8. Given a DG R-algebra A, a morphism of DG A-modules is an isomorphism if and only if it is injective and surjective.

The next exercise indicates that base change is a functor.

Exercise 5.22 Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$. (See Exercise 5.13.)

(a) Given an R-linear map $f\colon M \to N$, prove that the chain map $K \otimes_R f\colon K \otimes_R M \to K \otimes_R N$ is a morphism of DG K-modules. More generally, given a chain map of R-complexes $g\colon X \to Y$ and a DG R-algebra A, prove that the chain map $A \otimes_R g\colon A \otimes_R X \to A \otimes_R Y$ is a morphism of DG A-modules.
(b) Give an example showing that if g is a quasiisomorphism, then $A \otimes_R g$ need not be a quasiisomorphism. (Note that if A_i is is R-projective for each i (e.g., if $A = K$), then g being a quasiisomorphism implies that $A \otimes_R g$ is a quasiisomorphism by Fact 4.9.)
(c) Prove that the natural map $K \to R/(\mathbf{x})$ is a morphism of DG K-modules. More generally, prove that every morphism $A \to B$ of DG R-algebras is a morphism of DG A-modules, where B is a DG A-module via restriction of scalars.

Next, we use our running example to provide some morphisms of DG modules.

Example 5.23 We continue with the notation of Example 5.16.

Let $f\colon G \to \Sigma R$ be a morphism of DG U-modules:

$$\begin{tikzcd}
G = \arrow[d, "f"'] & \cdots \arrow[r, "1"] & Re_3 \arrow[r, "0"] & R1_2 \arrow[r, "1"] \arrow[d] & Re_1 \arrow[r, "0"] \arrow[d, "f_1"'] & R1_0 \arrow[r] \arrow[d] & 0 \\
\Sigma R & & & 0 \arrow[r] & R \arrow[r] & 0. &
\end{tikzcd}$$

Commutativity of the first square shows that $f = 0$. One can also see this from the following computation:

$$f_1(e_1) = f_1(e \cdot 1_0) = e f_0(1_0) = 0.$$

That is, the only morphism of DG U-modules $G \to \Sigma R$ is the zero-morphism. The same conclusion holds for any morphism $G \to \Sigma^{2n+1} R$ with $n \in \mathbb{Z}$.

On the other hand, for each $n \in \mathbb{N}$, every element $r \in R$ determines a morphism $g^{r,n}\colon G \to \Sigma^{2n} R$, via multiplication by r. For instance in the case $n = 1$:

$$\begin{tikzcd}
G = \arrow[d, "g^{r,1}"'] & \cdots \arrow[r, "1"] & Re_3 \arrow[r, "0"] \arrow[d] & R1_2 \arrow[r, "1"] \arrow[d, "g_2^{r,1}"', "=r\cdot"] & Re_1 \arrow[r, "0"] \arrow[d] & R1_0 \arrow[r] & 0 \\
\Sigma^2 R = & & 0 \arrow[r] & R \arrow[r] & 0. & &
\end{tikzcd}$$

Each square commutes, and the linearity condition is from the next computations:

$$\begin{aligned}
g_2^{r,1}(1 \cdot 1_2) &= g_2^{r,1}(1_2) = r = 1 \cdot r = 1 \cdot g_2^{r,1}(1_2) \\
g_2^{r,1}(e \cdot e_1) &= g_2^{r,1}(0) = 0 = e \cdot 0 = e \cdot g_1^{r,1}(e_1) \\
g_3^{r,1}(e \cdot 1_2) &= g_3^{r,1}(e_3) = 0 = e \cdot r = e \cdot g_2^{r,1}(1_2).
\end{aligned}$$

Further, the isomorphism $\mathrm{Hom}_R(R,R) \cong R$ shows that each morphism $G \to \Sigma^{2n}R$ is of the form $g^{r,n}$. Also, one checks readily that the map $G \xrightarrow{g^{u,0}} R$ is a quasiisomorphism for each unit u of R.

Truncations of DG Modules

The next operation allows us to swap a given DG module with a "shorter" one; see Exercise 5.27(b).

Definition 5.24 Let A be a DG R-algebra, and let M be a DG A-module. The *supremum* of M is

$$\sup(M) := \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(M) \neq 0\}.$$

Given an integer n, the *nth soft left truncation of M* is the complex

$$\tau(M)_{(\leqslant n)} := \quad 0 \to M_n / \mathrm{Im}(\partial^M_{n+1}) \to M_{n-1} \to M_{n-2} \to \cdots$$

with differential induced by ∂^M.

Example 5.25 We continue with the notation of Example 5.16. For each $n \geqslant 1$, set $i = \lfloor n/2 \rfloor$ and prove that

$$\tau(G)_{(\leqslant n)} = \quad 0 \to R1_{2i} \xrightarrow{1} Re_{2i-1} \xrightarrow{0} \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0.$$

Remark 5.26 Let P be a projective resolution of an R-module M, with P^+ denoting the augmented resolution as in Example 3.2. Then one has $\tau(P)_{(\leqslant 0)} \cong M$. In particular, P is a projective resolution of $\tau(P)_{(\leqslant 0)}$.

Exercise 5.27 Let A be a DG R-algebra, let M be a DG A-module, and let $n \in \mathbb{Z}$.

(a) Prove that the truncation $\tau(M)_{(\leqslant n)}$ is a DG A-module with the induced scalar multiplication, and the natural chain map $M \to \tau(M)_{(\leqslant n)}$ is a morphism of DG A-modules.
(b) Prove that the morphism from part (a) is a quasiisomorphism if and only if $n \geqslant \sup(M)$.

DG Algebra Resolutions

The following fact provides the final construction needed to give an initial sketch of the proof of Theorem 1.4.

Fact 5.28 Let $Q \to R$ be a ring epimorphism. Then there is a quasiisomorphism $A \xrightarrow{\simeq} R$ of DG Q-algebras such that each A_i is finitely generated and projective over R and $A_i = 0$ for $i > \mathrm{pd}_Q(R)$. See, e.g., [4, Proposition 2.2.8].

Definition 5.29 In Fact 5.28, the quasiisomorphism $A \xrightarrow{\simeq} R$ is a *DG algebra resolution* of R over Q.

Remark 5.30 When $\mathbf{y} \in Q$ is a Q-regular sequence, the Koszul complex $K^Q(\mathbf{y})$ is a DG algebra resolution of $Q/(\mathbf{y})$ over Q by Lemma 4.18 and Example 5.3. Section 6 contains other classical examples.

Example 5.31 Let Q be a ring, and consider an ideal $I \subsetneq Q$. Assume that the quotient $R := Q/I$ has $\mathrm{pd}_Q(R) \leqslant 1$. Then every projective resolution of R over Q of the form $A = (0 \to A_1 \to Q \to 0)$ has the structure of a DG algebra resolution.

We conclude this section with the beginning of the proof of Theorem 1.4. The rest of the proof is contained in 7.38 and 8.17.

5.32 (First part of the Proof of Theorem 1.4) There is a flat local ring homomorphism $R \to R'$ such that R' is complete with algebraically closed residue field, as in the proof of Theorem 2.13. Since there is a 1-1 function $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(R')$ by Fact 2.7, we can replace R with R' and assume without loss of generality that R is complete with algebraically closed residue field.

Since R is complete and local, Cohen's structure theorem provides a ring epimorphism $\tau\colon (Q,\mathfrak{n},k) \to (R,\mathfrak{m},k)$ where Q is a complete regular local ring such that $\mathfrak{m}$ and $\mathfrak{n}$ have the same minimal number of generators. Let $\mathbf{y} = y_1,\dots,y_n \in \mathfrak{n}$ be a minimal generating sequence for $\mathfrak{n}$, and set $\mathbf{x} = x_1,\dots,x_n \in \mathfrak{m}$ where $x_i := \tau(y_i)$. It follows that we have $K^R(\mathbf{x}) \cong K^Q(\mathbf{y}) \otimes_Q R$. Since Q is regular and $\mathbf{y}$ is a minimal generating sequence for $\mathfrak{n}$, the Koszul complex $K^Q(\mathbf{y})$ is a minimal Q-free resolution of k by Lemma 4.18.

Fact 5.28 provides a DG algebra resolution $A \xrightarrow{\simeq} R$ of R over Q. Note that $\mathrm{pd}_Q(R) < \infty$ since Q is regular. We consider the following diagram of morphisms of DG Q-algebras:

$$R \to K^R(\mathbf{x}) \cong K^Q(\mathbf{y}) \otimes_Q R \xleftarrow{\simeq} K^Q(\mathbf{y}) \otimes_Q A \xrightarrow{\simeq} k \otimes_Q A =: U. \tag{5.32.1}$$

The first map is from Exercise 5.6. The isomorphism is from the previous paragraph. The first quasiisomorphism comes from an application of $K^Q(\mathbf{y}) \otimes_Q -$ to the quasiisomorphism $R \xleftarrow{\simeq} A$, using Fact 4.9. The second quasiisomorphism comes from an application of $- \otimes_Q A$ to the quasiisomorphism $K^Q(\mathbf{y}) \xrightarrow{\simeq} k$. Note that $k \otimes_Q A$ is a finite dimensional DG k-algebra because of the assumptions on A.

We show in 7.38 below how this provides a diagram

$$\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R) \xrightarrow{\equiv} \mathfrak{S}(K^R(\mathbf{x})) \xleftarrow{\equiv} \mathfrak{S}(K^Q(\mathbf{y}) \otimes_Q A) \xrightarrow{\equiv} \mathfrak{S}(U) \tag{5.32.2}$$

where $\equiv$ identifies bijections of sets. We then show in 8.17 that $\mathfrak{S}(U)$ is finite, and it follows that $\mathfrak{S}_0(R)$ is finite, as desired.

6 Examples of Algebra Resolutions

Remark 5.30 and Example 5.31 provide constructions of DG algebra resolutions, in particular, for rings of projective dimension at most 1. The point of this section is to extend this to rings of projective dimension 2 and rings of projective dimension 3 determined by Gorenstein ideals.

Definition 6.1 Let I be an ideal of a local ring $(R,\mathfrak{m})$. The *grade* of I in R, denoted $\operatorname{grade}_R(I)$, is defined to be the length of the longest regular sequence of R contained in I. Equivalently, we have

$$\operatorname{grade}_R(I) := \min\left\{i \mid \operatorname{Ext}^i_R(R/I,R) \neq 0\right\}$$

and it follows that $\operatorname{grade}_R(I) \leqslant \operatorname{pd}_R(R/I)$. We say that I is *perfect* if $\operatorname{grade}_R(I) = \operatorname{pd}_R(R/I) < \infty$. In this case, $\operatorname{Ext}^i_R(R/I,R)$ is non-vanishing precisely when $i = \operatorname{pd}_R(R/I)$. If, in addition, this single non-vanishing cohomology module is isomorphic to R/I, then I is said to be *Gorenstein*.

Notation 6.2 Let A be a matrix[7] over R and $J,K \subset \mathbb{N}$. The submatrix of A obtained by deleting columns indexed by J and rows indexed by K is denoted A^J_K. We abbreviate $A^{\emptyset}_{\{i\}}$ as A_i, and so on. Let $I_n(A)$ be the ideal of R generated by the "$n\times n$ minors" of A, that is, the determinants of the $n \times n$ matrices of the form A^J_K.

Resolutions of length two

The following result, known as the Hilbert-Burch Theorem, provides a characterization of perfect ideals of grade two. It was first proven by Hilbert in 1890 in the case that R is a polynomial ring [27]; the more general statement was proven by Burch in 1968 [13, Theorem 5].

Theorem 6.3 ([13, 27]) *Let I be an ideal of the local ring $(R,\mathfrak{m})$.*

(a) *If $\operatorname{pd}_R(R/I) = 2$, then*

  (1) *there is a non-zerodivisor $a \in R$ such that R/I has a projective resolution of the form $0 \to R^n \xrightarrow{A} R^{n+1} \xrightarrow{B} R \to 0$ where B is the $1\times(n+1)$ matrix with ith column given by $(-1)^{i-1}a\det(A_i)$,*
  (2) *one has $I = aI_n(A)$, and*
  (3) *the ideal $I_n(A)$ is perfect of grade 2.*

(b) *Conversely, if A is an $(n+1)\times n$ matrix over R such that $\operatorname{grade}(I_n(A)) \geqslant 2$, then $R/I_n(A)$ has a projective resolution of the form $0 \to R^n \xrightarrow{A} R^{n+1} \xrightarrow{B} R \to 0$ where B is the $1\times(n+1)$ matrix with ith column given by $(-1)^{i-1}\det(A_i)$.*

[7] While much of the work in this section can be done basis-free, the formulations are somewhat more transparent when bases are specified and matrices are used to represent homomorphisms.

Exercise 6.4 Use Theorem 6.3 to build a grade two perfect ideal I in $R = k[\![x,y]\!]$.

Herzog [26] showed that Hilbert-Burch resolutions can be endowed with DG algebra structures.

Theorem 6.5 ([26]) *Given an $(n+1)\times n$ matrix A over R, let B be the $1\times(n+1)$ matrix with ith column given by $(-1)^{i-1}a\det(A_i)$ for some non-zerodivisor $a\in R$. Then the R-complex*

$$0 \to \bigoplus_{\ell=1}^{n} Rf_\ell \xrightarrow{A} \bigoplus_{\ell=1}^{n+1} Re_\ell \xrightarrow{B} R1 \to 0$$

has the structure of a DG R-algebra with the following multiplication relations:

(1) $e_i^2 = 0 = f_j f_k$ *and* $e_i f_j = 0 = f_j e_i$ *for all* i,j,k, *and*

(2) $e_i e_j = -e_j e_i = a\sum_{k=1}^{n}(-1)^{i+j+k}\det(A_{i,j}^k)f_k$ *for all* $1 \leqslant i < j \leqslant n+1$. [8]

Exercise 6.6 Verify the Leibniz rule for the product defined in Theorem 6.5.

Exercise 6.7 Using the ideal I from Exercise 6.4, build a (minimal) free resolution for R/I, then specify the relations giving this resolution a DG R-algebra structure.

Resolutions of length three

We turn our attention to resolutions of length three, first recalling needed machinery.

Definition 6.8 A square matrix A over R is *alternating* if it is skew-symmetric and has all 0's on its diagonal. Let A be an $n\times n$ alternating matrix over R. If n is even, then there is an element $\mathrm{Pf}(A)\in R$ such that $\mathrm{Pf}(A)^2 = \det(A)$. If n is odd then $\det(A) = 0$, so we set $\mathrm{Pf}(A) = 0$. The element $\mathrm{Pf}(A)$ is called the *Pfaffian* of A. (See, e.g., [11, Section 3.4] for more details.) We denote by $\mathrm{Pf}_{n-1}(A)$ the ideal of R generated by the submaximal Pfaffians of A, that is,

$$\mathrm{Pf}_{n-1}(A) := \big(\mathrm{Pf}(A_i^i) \mid 1 \leqslant i \leqslant n\big)R.$$

Example 6.9 Let $x,y,z\in R$. For the matrix $A = \left[\begin{smallmatrix} 0 & x \\ -x & 0\end{smallmatrix}\right]$, we have $\det(A) = x^2$, so $\mathrm{Pf}(A) = x$, and $\mathrm{Pf}_1(A) = 0$.

For the matrix $B = \begin{bmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0\end{bmatrix}$ we have $\det(B) = 0 = \mathrm{Pf}(B)$ and

$$\mathrm{Pf}_2(B) = (\mathrm{Pf}(\left[\begin{smallmatrix} 0 & x \\ -x & 0\end{smallmatrix}\right]), \mathrm{Pf}(\begin{bmatrix} 0 & y \\ -y & 0\end{bmatrix}), \mathrm{Pf}(\left[\begin{smallmatrix} 0 & z \\ -z & 0\end{smallmatrix}\right]))R = (x,y,z)R.$$

[8] Note that the sign in this expression differs from the one found in [4, Example 2.1.2].

Buchsbaum and Eisenbud [12] study the structure of resolutions of length three. Specifically, they characterize such resolutions and exhibit their DG structure.

Theorem 6.10 ([12, Theorem 2.1]) *Let I be an ideal of the local ring $(R,\mathfrak{m})$.*

(a) *If I is Gorenstein and $\mathrm{pd}_R(R/I) = 3$, then there is an odd integer $n \geqslant 3$ and an $n \times n$ matrix A over $\mathfrak{m}$ such that $I = \mathrm{Pf}_{n-1}(A)$ and R/I has a minimal free resolution of the form $0 \to R \xrightarrow{B^T} R^n \xrightarrow{A} R^n \xrightarrow{B} R \to 0$ where B is the $1 \times n$ matrix with ith column given by $(-1)^i\,\mathrm{Pf}(A_i)$.*

(b) *Conversely, if A is an $n \times n$ alternating matrix over $\mathfrak{m}$ such that $\mathrm{rank}(A) = n-1$, then n is odd and $\mathrm{grade}(\mathrm{Pf}_{n-1}(A)) \leqslant 3$; if $\mathrm{grade}(\mathrm{Pf}_{n-1}(A)) = 3$ then $R/\mathrm{Pf}_{n-1}(A)$ has a minimal free resolution of the form*

$$0 \to R \xrightarrow{B^T} R^n \xrightarrow{A} R^n \xrightarrow{B} R \to 0$$

where B is the $1 \times n$ matrix with ith column given by $(-1)^{i-1}\,\mathrm{Pf}(A_i)$.

It follows from Theorem 6.10 that the minimal number of generators of a grade-3 Gorenstein ideal must be odd.

Example 6.11 Let $R = k[\![x,y,z]\!]$ and consider the alternating matrix $B = \left[\begin{smallmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{smallmatrix}\right]$ from Example 6.9 with

$$I = \mathrm{Pf}_2(B) = (\mathrm{Pf}(\left[\begin{smallmatrix} 0 & x \\ -x & 0 \end{smallmatrix}\right]), \mathrm{Pf}(\left[\begin{smallmatrix} 0 & y \\ -y & 0 \end{smallmatrix}\right]), \mathrm{Pf}(\left[\begin{smallmatrix} 0 & z \\ -z & 0 \end{smallmatrix}\right]))R = (x,y,z)R.$$

Theorem 6.10 implies that a minimal free resolution of R/I over R is of the form

$$0 \to Rg \xrightarrow{\left[\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right]} Rf_1 \oplus Rf_2 \oplus Rf_3 \xrightarrow{\left[\begin{smallmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{smallmatrix}\right]} Re_1 \oplus Re_2 \oplus Re_3 \xrightarrow{[z\ -y\ x]} R1 \to 0.$$

(Compare this to the Koszul complex $K^R(x,-y,z)$.)

The next two examples of Buchsbaum and Eisenbud [12] illustrate that there exists, for any odd $n \geqslant 3$, a grade-3 Gorenstein ideal which is n-generated.

Example 6.12 ([12, Proposition 6.2]) Let $R = k[\![x,y,z]\!]$. For $n \geqslant 3$ odd, define M_n to be the $n \times n$ alternating matrix whose entries *above* the diagonal are

$$(M_n)_{ij} := \begin{cases} x & \text{if } i \text{ is odd and } j = i+1 \\ y & \text{if } i \text{ is even and } j = i+1 \\ z & \text{if } j = n-i+1 \\ 0 & \text{otherwise.} \end{cases}$$

For instance, we have

$$M_3 = \begin{bmatrix} 0 & x & z \\ -x & 0 & y \\ -z & -y & 0 \end{bmatrix} \qquad \mathrm{Pf}_2(M_3) = (x,y,z)R$$

$$M_5 = \begin{bmatrix} 0 & x & 0 & 0 & z \\ -x & 0 & y & z & 0 \\ 0 & -y & 0 & x & 0 \\ 0 & -z & -x & 0 & y \\ -z & 0 & 0 & -y & 0 \end{bmatrix} \qquad \mathrm{Pf}_4(M_5) = (y^2, xz, xy+z^2, yz, x^2)R.$$

Then $\mathrm{Pf}_{n-1}(M_n)$ is an n-generated Gorenstein ideal of grade 3 in R.

Example 6.13 ([12, Proposition 6.1]) Let M_n be a generic $(2n+1)\times(2n+1)$ alternating matrix over the ring $Q_n := R[\![x_{i,j} \mid 1 \leqslant i < j \leqslant 2n+1]\!]$. That is:

$$M_n := \begin{bmatrix} 0 & x_{1,2} & x_{1,3} & \cdots & x_{1,2n+1} \\ -x_{1,2} & 0 & x_{2,3} & \cdots & x_{2,2n+1} \\ -x_{1,3} & -x_{2,3} & 0 & & \\ \vdots & \vdots & & \ddots & \\ -x_{1,2n+1} & -x_{2,2n+1} & & & 0 \end{bmatrix}$$

Then, for every $n \geqslant 1$, the ideal $\mathrm{Pf}_{2n}(M_n)$ is Gorenstein of grade 3 in Q_n.

The next result provides an explicit description of the DG algebra structure on the resolution from Theorem 6.10. Note that our result is the more elemental version from [4, Example 2.1.3]. It is worth noting that Buchsbaum and Eisenbud show that every free resolution over R of the form $0 \to F_3 \to F_2 \to F_1 \to R \to 0$ has the structure of a DG R-algebra.

Theorem 6.14 ([12, Theorem 4.1]) *Let A be an $n \times n$ alternating matrix over R, and let B be the $1 \times n$ matrix whose ith column is given by $(-1)^{i-1}\,\mathrm{Pf}(A_i^i)$. Then the graded R-complex*

$$0 \to Rg \xrightarrow{B^{\mathsf{T}}} \bigoplus_{\ell=1}^{n} Rf_\ell \xrightarrow{A} \bigoplus_{\ell=1}^{n} Re_\ell \xrightarrow{B} R1 \to 0$$

admits the structure of a DG R-algebra, with the following products:

(1) $e_i^2 = 0$ *and* $e_i f_j = f_j e_i = \delta_{ij} g$ *for all* $1 \leqslant i, j \leqslant n$, *and*

(2) $e_i e_j = -e_j e_i = \sum_{k=1}^{n} (-1)^{i+j+k} \rho_{ijk}\,\mathrm{Pf}(A_{ijk}^{ijk}) f_k$ *for all* $1 \leqslant i \neq j \leqslant n$, *where* $\rho_{ijk} = -1$ *whenever* $i < k < j$, *and* $\rho_{ijk} = 1$ *otherwise.*

Exercise 6.15 Let $R = k[x,y,z]$ and consider the graded R-complex given by

$$0 \to Rg \xrightarrow{\left[\begin{smallmatrix} z \\ -y \\ x \end{smallmatrix}\right]} Rf_1 \oplus Rf_2 \oplus Rf_3 \xrightarrow{\left[\begin{smallmatrix} 0 & x & y \\ -x & 0 & z \\ -y & -z & 0 \end{smallmatrix}\right]} Re_1 \oplus Re_2 \oplus Re_3 \xrightarrow{[z\ -y\ x]} R1 \to 0$$

Using Theorem 6.14, write the product relations that give this complex the structure of a DG R-algebra.

Longer resolutions

In general, resolutions of length greater than 3 are not guaranteed to possess a DG algebra structure, as the next example of Avramov shows.

Example 6.16 ([4, Theorem 2.3.1]) Consider the local ring $R = k[\![w,x,y,z]\!]$. Then the minimal free resolutions over R of the quotients $R/(w^2,wx,xy,yz,z^2)R$ and $R/(w^2,wx,xy,yz,z^2,wy^6,x^7,x^6z,y^7)R$ do not admit DG R-algebra structures.

On the other hand, resolutions of ideals I with $\mathrm{pd}_R(R/I) \geqslant 4$ that are sufficiently nice do admit DG algebra structures. For instance, Kustin and Miller prove the following in [28, Theorem] and [29, 4.3 Theorem].

Example 6.17 ([28, 29]) Let I be a Gorenstein ideal of a local ring R. If $\mathrm{pd}_R(R/I) = 4$, then the minimal R-free resolution of R/I has the structure of a DG R-algebra.

7 DG Algebras and DG Modules II

In this section, we describe the notions needed to define semidualizing DG modules and to explain some of their base-change properties. This includes a discussion of two types of Ext for DG modules. The section concludes with another piece of the proof of Theorem 1.4; see 8.17.

Convention 7.1 Throughout this section, A is a DG R-algebra, and L, M, and N are DG A-modules.

Hom for DG Modules

The semidualizing property for R-modules is defined in part by a Hom condition, so we begin our treatment of the DG-version with Hom.

Definition 7.2 Given an integer i, a *DG A-module homomorphism of degree n* is an element $f \in \mathrm{Hom}_R(M,N)_n$ such that $f_{i+j}(am) = (-1)^{ni} a f_j(m)$ for all $a \in A_i$ and $m \in M_j$. The graded submodule of $\mathrm{Hom}_R(M,N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_A(M,N)$.

Part (b) of the next exercise gives another hint of the semidualizing property for DG modules.

Exercise 7.3

(a) Prove that $\operatorname{Hom}_A(M,N)$ is a DG A-module via the action

$$(af)_j(m) := a(f_j(m)) = (-1)^{|a||f|} f_{j+|a|}(am)$$

and using the differential from $\operatorname{Hom}_R(M,N)$.

(b) Prove that for each $a \in A$ the multiplication map $\mu^{M,a}\colon M \to M$ given by $m \mapsto am$ is a homomorphism of degree $|a|$.

(c) Prove that $f \in \operatorname{Hom}_A(M,N)_0$ is a morphism if and only if it is a cycle, that is, if and only if $\partial_0^{\operatorname{Hom}_A(M,N)}(f) = 0$.

Example 7.4 We continue with the notation of Example 5.16. From computations like those in Example 5.23, it follows that $\operatorname{Hom}_U(G,R)$ has the form

$$\operatorname{Hom}_U(G,R) = \quad 0 \to R \to 0 \to R \to 0 \to \cdots$$

where the copies of R are in even non-positive degrees. Multiplication by e is 0 on $\operatorname{Hom}_U(G,R)$, by degree considerations, and multiplication by 1 is the identity.

Next, we give an indication of the functoriality of $\operatorname{Hom}_A(N,-)$ and $\operatorname{Hom}_A(-,N)$.

Definition 7.5 Given a morphism $f\colon L \to M$ of DG A-modules, we define the map $\operatorname{Hom}_A(N,f)\colon \operatorname{Hom}_A(N,L) \to \operatorname{Hom}_A(N,M)$ as follows: each sequence $\{g_p\} \in \operatorname{Hom}_A(N,L)_n$ is mapped to $\{f_{p+n}g_p\} \in \operatorname{Hom}_A(N,M)_n$. Similarly, define the map $\operatorname{Hom}_A(f,N)\colon \operatorname{Hom}_A(M,N) \to \operatorname{Hom}_A(L,N)$ by the formula $\{g_p\} \mapsto \{g_p f_p\}$.

Remark 7.6 We do not use a sign-change in this definition because $|f| = 0$.

Exercise 7.7 Given a morphism $f\colon L \to M$ of DG A-modules, prove that the maps $\operatorname{Hom}_A(N,f)$ and $\operatorname{Hom}_A(f,N)$ are well-defined morphisms of DG A-modules.

Tensor Product for DG Modules

As with modules and complexes, we use the tensor product to base change DG modules along a morphism of DG algebras.

Definition 7.8 The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is generated over R by the elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Exercise 7.9 Prove that the tensor product $M \otimes_A N$ is a DG A-module via the scalar multiplication

$$a(m \otimes n) := (am) \otimes n = (-1)^{|a||m|} m \otimes (an).$$

The next exercises describe base change and some canonical isomorphisms for DG modules.

Exercise 7.10 Let $A \to B$ be a morphism of DG R-algebras. Prove that $B \otimes_A M$ has the structure of a DG B-module by the action $b(b' \otimes m) := (bb') \otimes m$. Prove that this structure is compatible with the DG A-module structure on $B \otimes_A M$ via restriction of scalars.

Exercise 7.11 Verify the following isomorphisms of DG A-modules:

$\mathrm{Hom}_A(A, L) \cong L$	Hom cancellation
$A \otimes_A L \cong L$	tensor cancellation
$L \otimes_A M \cong M \otimes_A L$	tensor commutativity

In particular, there are DG A-module isomorphisms $\mathrm{Hom}_A(A,A) \cong A \cong A \otimes_A A$.

Fact 7.12 There is a natural "Hom tensor adjointness" DG A-module isomorphism $\mathrm{Hom}_A(L \otimes_A M, N) \cong \mathrm{Hom}_A(M, \mathrm{Hom}_A(L, N))$.

Next, we give an indication of the functoriality of $N \otimes_A -$ and $- \otimes_A N$.

Definition 7.13 Given a morphism $f\colon L \to M$ of DG A-modules, we define the map $N \otimes_A f\colon N \otimes_A L \to N \otimes_A M$ by the formula $z \otimes y \mapsto z \otimes f(y)$. Define the map $f \otimes_A N\colon L \otimes_A N \to M \otimes_A N$ by the formula $x \otimes z \mapsto f(x) \otimes z$.

Remark 7.14 We do not use a sign-change in this definition because $|f| = 0$.

Exercise 7.15 Given a morphism $f\colon L \to M$ of DG A-modules, prove that the maps $N \otimes_A f$ and $f \otimes_R N$ are well-defined morphisms of DG A-modules.

Semifree Resolutions

Given the fact that the semidualizing property includes an Ext-condition, it should come as no surprise that we need a version of free resolutions in the DG setting.

Definition 7.16 A subset E of L is called a *semibasis* if it is a basis of the underlying $A^\natural$-module $L^\natural$. If L is bounded below, then L is called *semi-free* if it has a semibasis.[9] A *semi-free resolution* of a DG A-module M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of DG A-modules such that F is semi-free.

The next exercises and example give some semi-free examples to keep in mind.

Exercise 7.17 Prove that a semi-free DG R-module is simply a bounded below complex of free R-modules. Prove that each free resolution F of an R-module M gives rise to a semi-free resolution $F \xrightarrow{\simeq} M$; see Exercise 3.10.

[9] As is noted in [6], when L is not bounded below, the definition of "semi-free" is more technical. However, our results do not require this level of generality, so we focus only on this case.

Exercise 7.18 Prove that M is exact (as an R-complex) if and only if $0 \xrightarrow{\simeq} M$ is a semi-free resolution. Prove that the DG A-module $\Sigma^n A$ is semi-free for each $n \in \mathbb{Z}$, as is $\bigoplus_{n \geqslant n_0} \Sigma^n A^{\beta_n}$ for all $n_0 \in \mathbb{Z}$ and $\beta_n \in \mathbb{N}$.

Exercise 7.19 Let $\mathbf{x} = x_1, \cdots, x_n \in R$, and set $K = K^R(\mathbf{x})$.

(a) Given a bounded below complex F of free R-modules, prove that the complex $K \otimes_R F$ is a semi-free DG K-module.
(b) If $F \xrightarrow{\simeq} M$ is a free resolution of an R-module M, prove that $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a semi-free resolution of the DG K-module $K \otimes_R M$. More generally, if $F \xrightarrow{\simeq} M$ is a semi-free resolution of a DG R-module M, prove that $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a semi-free resolution of the DG K-module $K \otimes_R M$. See Fact 4.9.

Example 7.20 In the notation of Example 5.16, the natural map $g^{1,0} \colon G \to R$ is a semi-free resolution of R over U; see Example 5.23. The following display indicates why G is semi-free over U, that is, why $G^\natural$ is free over $U^\natural$:

$$\begin{aligned} U = {} & \quad 0 \to Re \xrightarrow{0} R1 \to 0 \\ U^\natural = {} & \quad Re \bigoplus R1 \\ G = {} & \cdots \xrightarrow{1} Re_3 \xrightarrow{0} R1_2 \xrightarrow{1} Re_1 \xrightarrow{0} R1_0 \to 0 \\ G^\natural = {} & \cdots (Re_3 \bigoplus R1_2) \bigoplus (Re_1 \bigoplus R1_0). \end{aligned}$$

The next item compares to Remark 5.26.

Remark 7.21 If L is semi-free, then the natural map $L \to \tau(L)_{(\leqslant n)}$ is a semi-free resolution for each $n \geqslant \sup(L)$.

The next facts contain important existence results for semi-free resolutions. Notice that the second paragraph applies when A is a Koszul complex over R or is finite dimensional over a field, by Exercise 5.10.

Fact 7.22 The DG A-module M has a semi-free resolution if and only if $\mathrm{H}_i(M) = 0$ for $i \ll 0$, by [6, Theorem 2.7.4.2].

Assume that A is noetherian, and let j be an integer. Assume that each module $\mathrm{H}_i(M)$ is finitely generated over $\mathrm{H}_0(A)$ and that $\mathrm{H}_i(M) = 0$ for $i < j$. Then M has a semi-free resolution $F \xrightarrow{\simeq} M$ such that $F^\natural \cong \bigoplus_{i=j}^{\infty} \Sigma^i (A^\natural)^{\beta_i}$ for some integers β_i, and so $F_i = 0$ for all $i < j$; see [1, Proposition 1]. In particular, homologically finite DG A-modules admit "degree-wise finite, bounded below" semi-free resolutions.

Fact 7.23 Assume that L and M are semi-free. If there is a quasiisomorphism $L \xrightarrow{\simeq} M$, then there is also a quasiisomorphism $M \xrightarrow{\simeq} L$ by [4, Proposition 1.3.1].

The previous fact explains why the next relations are symmetric. The fact that they are reflexive and transitive are straightforward to verify.

Definition 7.24 Two semi-free DG A-modules L and M are *quasiisomorphic* if there is a quasiisomorphism $L \xrightarrow{\simeq} M$; this equivalence relation is denoted by the symbol $\simeq$. Two semi-free DG A-modules L and M are *shift-quasiisomorphic* if there is an integer m such that $L \simeq \Sigma^m M$; this equivalence relation is denoted by $\sim$.

Semidualizing DG Modules

For Theorem 1.4, we use a version of Christensen and Sather-Wagstaff's notion of semidualizing DG modules from [16], defined next.

Definition 7.25 The *homothety morphism* $\chi^A_M \colon A \to \operatorname{Hom}_A(M,M)$ is given by the formula $(\chi^A_M)_{|a|}(a) := \mu^{M,a}$, i.e., $(\chi^A_M)_{|a|}(a)_{|m|}(m) = am$.

Assume that A is noetherian. Then M is a *semidualizing* DG A-module if M is homologically finite and semi-free such that $\chi^A_M \colon A \to \operatorname{Hom}_A(M,M)$ is a quasiisomorphism. Let $\mathfrak{S}(A)$ denote the set of shift-quasiisomorphism classes of semidualizing DG A-modules, that is, the set of equivalence classes of semidualizing DG A-modules under the relation $\sim$ from Definition 7.24.

Exercise 7.26 Prove that the homothety morphism $\chi^A_M \colon A \to \operatorname{Hom}_A(M,M)$ is a well-defined morphism of DG A-modules.

The following fact explains part of diagram (5.32.2).

Fact 7.27 Let M be an R-module with projective resolution P. Then Fact 3.18 shows that M is a semidualizing R-module if and only if P is a semidualizing DG R-module. It follows that we have an injection $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}(R)$.

The next example justifies our focus on shift-quasiisomorphism classes of semidualizing DG A-modules.

Example 7.28 Let B and C be semi-free DG A-modules such that $B \sim C$. Then B is semidualizing over A if and only if C is semidualizing over A. The point is the following. The condition $B \simeq \Sigma^i C$ tells us that B is homologically finite if and only if $\Sigma^i C$ is homologically finite, that is, if and only if C is homologically finite. Fact 7.23 provides a quasiisomorphism $B \underset{\simeq}{\xrightarrow{f}} \Sigma^i C$. Thus, there is a commutative diagram of morphisms of DG A-modules:

$$\begin{tikzcd}
\operatorname{Hom}_A(C,C) \arrow[dr, "\cong"'] & A \arrow[l, "\chi^A_C"'] \arrow[r, "\chi^A_B"] \arrow[d, "\chi^A_{\Sigma^i C}"'] & \operatorname{Hom}_A(B,B) \arrow[d, "\simeq"', "\operatorname{Hom}_A(B,f)"] \\
& \operatorname{Hom}_A(\Sigma^i C, \Sigma^i C) \arrow[r, "\operatorname{Hom}_A(f,\Sigma^i C)", "\simeq"'] & \operatorname{Hom}_A(B, \Sigma^i C).
\end{tikzcd}$$

The unspecified isomorphism follows from a bookkeeping exercise.[10] The morphisms $\mathrm{Hom}_A(f,C)$ and $\mathrm{Hom}_A(B,f)$ are quasiisomorphisms by [4, Propositions 1.3.2 and 1.3.3] because B and $\Sigma^i C$ are semi-free and f is a quasiisomorphism. It follows that χ_B^A is a quasiisomorphism if and only if $\chi_{\Sigma^i C}^A$ is a quasiisomorphism if and only if χ_C^A is a quasiisomorphism.

The following facts explain other parts of (5.32.2).

Fact 7.29 Assume that $(R,\mathfrak{m})$ is local. Fix a list of elements $\mathbf{x} \in \mathfrak{m}$ and set $K = K^R(\mathbf{x})$. Base change $K \otimes_R -$ induces an injective map $\mathfrak{S}(R) \hookrightarrow \mathfrak{S}(K)$ by [16, A.3. Lemma]; if R is complete, then this map is bijective by [33, Corollary 3.10].

Fact 7.30 Let $\varphi\colon A \xrightarrow{\simeq} B$ be a quasiisomorphism of noetherian DG R-algebras. Base change $B \otimes_A -$ induces a bijection from $\mathfrak{S}(A)$ to $\mathfrak{S}(B)$ by [32, Lemma 2.22(c)].

Ext for DG Modules

One subtlety in the proof of Fact 7.29 is found in the behavior of Ext for DG modules, which we describe next.

Definition 7.31 Given a semi-free resolution $F \xrightarrow{\simeq} M$, for each integer i we set $\mathrm{Ext}_A^i(M,N) := \mathrm{H}_{-i}(\mathrm{Hom}_A(F,N))$.[11]

The next two items are included in our continued spirit of providing perspective.

Exercise 7.32 Given R-modules M and N, prove that the module $\mathrm{Ext}_R^i(M,N)$ defined in 7.31 is the usual $\mathrm{Ext}_R^i(M,N)$; see Exercise 7.17.

Example 7.33 In the notation of Example 5.16, Examples 7.4 and 7.20 imply

$$\mathrm{Ext}_U^i(R,R) = \mathrm{H}_{-i}(\mathrm{Hom}_U(G,R)) = \begin{cases} R & \text{if } i \geqslant 0 \text{ is even} \\ 0 & \text{otherwise.} \end{cases}$$

Contrast this with the equality $\mathrm{Ext}_{U^\natural}^i(R,R) = R$ for all $i \geqslant 0$. This shows that U is fundamentally different from $U^\natural \cong R[X]/(X^2)$, even though U is obtained using a trivial differential on $R[X]/(X^2)$ with the natural grading.

The next result compares with the fact that $\mathrm{Ext}_R^i(M,N)$ is independent of the choice of free resolution when M and N are modules.

Fact 7.34 For each index i, the module $\mathrm{Ext}_A^i(M,N)$ is independent of the choice of semi-free resolution of M by [4, Proposition 1.3.3].

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10 The interested reader may wish to show how this isomorphism is defined and to check the commutativity of the diagram. If this applies to you, make sure to mind the signs.

11 One can also define $\mathrm{Tor}_i^R(M,N) := \mathrm{H}_i(F \otimes_A N)$, but we do not need this here.

Remark 7.35 An important fact about $\mathrm{Ext}^1_R(M,N)$ for R-modules M and N is the following: the elements of $\mathrm{Ext}^1_R(M,N)$ are in bijection with the equivalence classes of short exact sequences (i.e., "extensions") of the form $0 \to N \to X \to M \to 0$. For DG modules over a DG R-algebra A, things are a bit more subtle.

Given DG A-modules M and N, one defines the notion of a short exact sequence of the form $0 \to N \to X \to M \to 0$ in the naive way: the arrows are morphisms of DG A-modules such that for each $i \in \mathbb{Z}$ the sequence $0 \to N_i \to X_i \to M_i \to 0$ is exact. One defines an equivalence relation on the set of short exact sequences of this form (i.e., "extensions") in the natural way: two extensions $0 \to N \xrightarrow{f} X \xrightarrow{g} M \to 0$ and $0 \to N \xrightarrow{f'} X' \xrightarrow{g'} M \to 0$ are equivalent if there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "f"] \arrow[d, "="'] & X \arrow[r, "g"] \arrow[d, "h"'] & M \arrow[r] \arrow[d, "="'] & 0 \\ 0 \arrow[r] & N \arrow[r, "f'"] & X' \arrow[r, "g'"] & M \arrow[r] & 0 \end{tikzcd}$$

of morphisms of DG A-modules. Let $\mathrm{YExt}^1_A(M,N)$ denote the set of equivalence classes of such extensions. (The "Y" is for "Yoneda".) As with R-modules, one can define an abelian group structure on $\mathrm{YExt}^1_A(M,N)$. However, in general one has $\mathrm{YExt}^1_A(M,N) \ncong \mathrm{Ext}^1_A(M,N)$, even when $A = R$, as the next example shows.

Example 7.36 Let $R = k[\![X]\!]$, and consider the following exact sequence of DG R-modules, i.e., exact sequence of R-complexes:

$$0 \longrightarrow \underline{R} \longrightarrow \underline{R} \longrightarrow \underline{k} \longrightarrow 0$$

$$\begin{tikzcd} & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\ 0 \arrow[r] & R \arrow[r, "X"] \arrow[d, "1"'] & R \arrow[r] \arrow[d, "1"'] & k \arrow[r] \arrow[d, "1"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "X"] \arrow[d] & R \arrow[r] \arrow[d] & k \arrow[r] \arrow[d] & 0 \\ & 0 & 0 & 0. & \end{tikzcd}$$

This sequence does not split over R (it is not even degree-wise split) so it gives a non-trivial class in $\mathrm{YExt}^1_R(\underline{k},\underline{R})$, and we conclude that $\mathrm{YExt}^1_R(\underline{k},\underline{R}) \neq 0$. On the other hand, $\underline{k}$ is homologically trivial, so we have $\mathrm{Ext}^1_R(\underline{k},\underline{R}) = 0$ since $0 \xrightarrow{\simeq} \underline{k}$ is a semi-free resolution.

For our proof of Theorem 1.4, the following connection between Ext and YExt is quite important; see [32, Corollary 3.8 and Proposition 3.12].

Fact 7.37 If L is semi-free, then we have $\mathrm{YExt}^1_A(L,M) \cong \mathrm{Ext}^1_A(L,M)$; if furthermore $\mathrm{Ext}^1_R(L,L) = 0$, then for each $n \geqslant \sup(L)$, one has

$$\mathrm{YExt}^1_A(L,L) = 0 = \mathrm{YExt}^1_A(\tau(L)_{(\leqslant n)}, \tau(L)_{(\leqslant n)}).$$

We conclude this section with the second part of the proof of Theorem 1.4. The rest of the proof is contained in 8.17.

7.38 (Second part of the proof of Theorem 1.4) We continue with the notation established in 5.32. The properties of diagram (5.32.2) follow from diagram (5.32.1) because of Facts 7.29 and 7.30. Thus, it remains to show that $\mathfrak{S}(k \otimes_Q A)$ is finite. This is shown in 8.17.

8 A Version of Happel's Result for DG Modules

This section contains the final steps of the proof of Theorem 1.4; see 8.17. The idea, from [2, 22, 25, 42] is a bold one: use algebraic geometry to study all possible module structures on a fixed set. A simple case of this is in Example 5.17. We begin with some notation for use throughout this section.

Notation 8.1 Let F be an algebraically closed field, and let

$$U := (0 \to U_q \xrightarrow{\partial^U_q} U_{q-1} \xrightarrow{\partial^U_{q-1}} \cdots \xrightarrow{\partial^U_1} U_0 \to 0)$$

be a finite-dimensional DG F-algebra. Let $\dim_F(U_i) = n_i$ for $i = 0, \ldots, q$. Let

$$W := \bigoplus_{i=0}^{s} W_i$$

be a graded F-vector space with $r_i := \dim_F(W_i)$ for $i = 0, \ldots, s$.

A DG U-module structure on W consists of two pieces of data. First, we need a differential ∂. Second, once the differential ∂ has been chosen, we need a scalar multiplication μ. Let $\mathrm{Mod}^U(W)$ denote the set of all ordered pairs (∂, μ) making W into a DG U-module. Let $\mathrm{End}_F(W)_0$ denote the set of F-linear endomorphisms of W that are homogeneous of degree 0. Let $\mathrm{GL}(W)_0$ denote the set of F-linear automorphisms of W that are homogeneous of degree 0, that is, the invertible elements of $\mathrm{End}_F(W)_0$.

We next describe a geometric structure on $\mathrm{Mod}^U(W)$, as in Example 5.17.

Discussion 8.2 We work in the setting of Notation 8.1.

A differential ∂ on W is an element of the graded vector space $\mathrm{Hom}_F(W,W)_{-1} = \bigoplus_{i=0}^{s} \mathrm{Hom}_F(W_i, W_{i-1})$ such that $\partial\partial = 0$. The vector space $\mathrm{Hom}_F(W_i, W_{i-1})$ has dimension $r_i r_{i-1}$, so the map ∂ corresponds to an element of the affine space $\mathbb{A}^d_F$ where $d := \sum_i r_i r_{i-1}$. The vanishing condition $\partial\partial = 0$ is equivalent to the entries of the matrices representing ∂ satisfying certain fixed homogeneous quadratic polynomial equations over F. Hence, the set of all differentials on W is a Zariski-closed subset of $\mathbb{A}^d_F$.

Once the differential ∂ has been chosen, a scalar multiplication μ is in particular a cycle in $\mathrm{Hom}_F(U\otimes_F W,W)_0=\bigoplus_{i,j}\mathrm{Hom}_F(U_i\otimes_F W_j,W_{i+j})$. For all i,j, the vector space $\mathrm{Hom}_F(U_i\otimes_F W_j,W_{i+j})$ has dimension $n_ir_jr_{i+j}$, so the map μ corresponds to an element of the affine space $\mathbb{A}_F^{d'}$ where $d':=\sum_{i,j}n_ir_jr_{i+j}$. The condition that μ be an associative, unital cycle is equivalent to the entries of the matrices representing ∂ and μ satisfying certain fixed polynomials over F. Thus, the set $\mathrm{Mod}^U(W)$ is a Zariski-closed subset of $\mathbb{A}_F^d\times\mathbb{A}_F^{d'}\cong\mathbb{A}_F^{d+d'}$.

Example 8.3 We continue with the notation of Example 5.17. In this example, we have $\mathrm{Mod}^U(W)=\{1\}$ (representing the non-trivial scalar multiplication by 1) and $\mathrm{Mod}^U(W')=\{(x_1,x_0)\in F^2\mid x_1x_0=0\}$. Re-writing F^n as $\mathbb{A}_F^n$, we see that $\mathrm{Mod}^U(W)$ is a single point in $\mathbb{A}_F^1$ and $\mathrm{Mod}^U(W')$ is the union of the two coordinate axes $V(x_1x_0)=V(x_0)\cup V(x_1)$.

Exercise 8.4 Continue with the notation of Example 5.17. Write out the coordinates and equations describing $\mathrm{Mod}^U(W'')$ and $\mathrm{Mod}^U(W''')$ where

$$\begin{aligned} W''= &\quad 0\bigoplus Fw_2\bigoplus Fw_1\bigoplus Fw_0\bigoplus 0\\ W'''= &\quad 0\bigoplus Fz_2\bigoplus(Fz_{1,1}\bigoplus Fz_{1,2})\bigoplus Fz_0\bigoplus 0. \end{aligned}$$

For scalar multiplication, note that since multiplication by 1 is already determined by the F-vector space structure, we only need to worry about multiplication by e which maps $W_i''\to W_{i+1}''$ and $W_i'''\to W_{i+1}'''$ for $i=0,1,2$.

We next describe a geometric structure on the set $\mathrm{GL}(W)_0$.

Discussion 8.5 We work in the setting of Notation 8.1.

A map $\alpha\in\mathrm{GL}(W)_0$ is an element of the graded vector space $\mathrm{Hom}_F(W,W)_0=\bigoplus_{i=0}^s\mathrm{Hom}_F(W_i,W_i)$ with a multiplicative inverse. The vector space $\mathrm{Hom}_F(W_i,W_i)$ has dimension r_i^2, so the map α corresponds to an element of the affine space $\mathbb{A}_F^e$ where $e:=\sum_i r_i^2$. The invertibility of α is equivalent to the invertibility of each “block” $\alpha_i\in\mathrm{Hom}_F(W_i,W_i)$, which is an open condition defined by the non-vanishing of the determinant polynomial. Thus, the set $\mathrm{GL}(W)_0$ is a Zariski-open subset of $\mathbb{A}_F^e$, so it is smooth over F.

Alternately, one can view $\mathrm{GL}(W)_0$ as the product $\mathrm{GL}(W_0)\times\cdots\times\mathrm{GL}(W_s)$. Since each $\mathrm{GL}(W_i)$ is an algebraic group smooth over F, it follows that $\mathrm{GL}(W)_0$ is also an algebraic group that is smooth over F.

Example 8.6 We continue with the notation of Example 5.17. It is straightforward to show that

$$\begin{aligned} \mathrm{End}_F(W)_0&=\mathrm{Hom}_F(Fv_0,Fv_0)\cong F=\mathbb{A}_F^1\\ \mathrm{GL}_F(W)_0&=\mathrm{Aut}_F(Fv_0)\cong F^\times=U_x\subset\mathbb{A}_F^1\\ \mathrm{End}_F(W')_0&=\mathrm{Hom}_F(F\eta_1,F\eta_1)\bigoplus\mathrm{Hom}_F(F\eta_0,F\eta_0)\cong F\times F=\mathbb{A}_F^2\\ \mathrm{GL}_F(W')_0&=\mathrm{Aut}_F(F\eta_1)\bigoplus\mathrm{Aut}_F(F\eta_0)\cong F^\times\times F^\times=U_{x_1x_0}\subset\mathbb{A}_F^2. \end{aligned}$$

Here U_x is the subset $\mathbb{A}^1_F \smallsetminus V(x)$, and $U_{x_1x_0} = \mathbb{A}^2_F \smallsetminus V(x_1x_0)$.

Exercise 8.7 With the notation of Example 5.17. Give coordinates and equations describing $\mathrm{GL}^U(W'')_0$ and $\mathrm{GL}^U(W''')_0$ where W'' and W''' are from Exercise 8.4.

Next, we describe an action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$.

Discussion 8.8 We work in the setting of Notation 8.1.

Let $\alpha \in \mathrm{GL}(W)_0$. For every $(\partial,\mu) \in \mathrm{Mod}^U(W)$, we define $\alpha\cdot(\partial,\mu) := (\widetilde{\partial},\widetilde{\mu})$, where $\widetilde{\partial}_i := \alpha_{i-1}\circ\partial_i\circ\alpha_i^{-1}$ and $\widetilde{\mu}_{i+j} := \alpha_{i+j}\circ\mu_{i+j}\circ(U\otimes_F \alpha_j^{-1})$. For the multiplication, this defines a new multiplication

$$u_i \cdot_\alpha w_j := \alpha_{i+j}(u_i\cdot\alpha_j^{-1}(w_j))$$

where $\cdot$ is multiplication given by μ, as in Discussion 8.2: $u_i\cdot w_j := \mu_{i+j}(u_i\otimes w_j)$. Note that this leaves multiplication by 1_A unaffected:

$$1_A\cdot_\alpha w_j = \alpha_j(1_A\cdot\alpha_j^{-1}(w_j)) = \alpha_j(\alpha_j^{-1}(w_j)) = w_j.$$

It is routine to show that the ordered pair $(\widetilde{\partial},\widetilde{\mu})$ describes a DG U-module structure for W, that is, we have $\alpha\cdot(\partial,\mu) := (\widetilde{\partial},\widetilde{\mu}) \in \mathrm{Mod}^U(W)$. From the definition of $\alpha\cdot(\partial,\mu)$, it follows readily that this describes a $\mathrm{GL}(W)_0$-action on $\mathrm{Mod}^U(W)$.

Example 8.9 Continue with the notation of Example 5.17.

In this case, the only DG U-module structure on W is the trivial one $(\partial,\mu) = (0,0)$, so we have $\alpha\cdot(\partial,\mu) = (\partial,\mu)$ for all $\alpha \in \mathrm{GL}(W)_0$.

The action of $\mathrm{GL}(W')_0$ on $\mathrm{Mod}^U(W')$ is a bit more interesting. Let $x_0, x_1 \in F$ such that $x_0x_1 = 0$, as in Example 5.17. Identify $\mathrm{GL}_F(W')_0$ with $F^\times \times F^\times$, as in Example 8.3, and let $\alpha \in \mathrm{GL}_F(W')_0$ be given by the ordered pair $(y_1,y_0) \in F^\times\times F^\times$. The differential $\widetilde{\partial}$ is defined so that the following diagram commutes.

$$\begin{tikzcd}
\partial: & 0 \arrow[r] & F\eta_1 \arrow[r, "x_1"] \arrow[d, "y_1"'] & F\eta_0 \arrow[r] \arrow[d, "y_0"'] & 0 \\
\widetilde{\partial}: & 0 \arrow[r] & F\widetilde{\eta}_1 \arrow[r, "\widetilde{x}_1"] & F\widetilde{\eta}_0 \arrow[r] & 0
\end{tikzcd}$$

so we have $\widetilde{\partial}_1(\widetilde{\eta}_1) = y_0x_1y_1^{-1}\widetilde{\eta}_0$, i.e., $\widetilde{x}_1 = y_0x_1y_1^{-1}$.

Since multiplication by 1 is already determined, and we have $e\cdot_\alpha\widetilde{\eta}_1 = 0$ because of degree considerations, we only need to understand $e\cdot_\alpha\widetilde{\eta}_0$. From Discussion 8.8, this is given by

$$\begin{aligned}
e\cdot_\alpha\widetilde{\eta}_0 &= \alpha_1(e\cdot\alpha_0^{-1}(\widetilde{\eta}_0)) = \alpha_1(e\cdot y_0^{-1}\eta_0) = y_0^{-1}\alpha_1(e\cdot\eta_0)\\
&= y_0^{-1}\alpha_1(x_0\eta_1) = y_0^{-1}y_1x_0\widetilde{\eta}_1.
\end{aligned}$$

Exercise 8.10 Continue with the notation of Example 5.17. Using the solutions to Exercises 8.4 and 8.7 describe the actions of $\mathrm{GL}(W'')_0$ and $\mathrm{GL}(W''')_0$ on $\mathrm{Mod}^U(W'')$ and $\mathrm{Mod}^U(W''')$, respectively, as in the previous example.

Next, we describe some properties of the action from Discussion 8.8 that indicate a deeper connection between the algebra and geometry.

Discussion 8.11 We work in the setting of Notation 8.1.

Let $\alpha \in \mathrm{GL}(W)_0$. For every $(\partial,\mu) \in \mathrm{Mod}^U(W)$, let $\alpha \cdot (\partial,\mu) := (\widetilde{\partial},\widetilde{\mu})$ be as in Discussion 8.8. It is straightforward to show that a map α gives a DG U-module isomorphism $(W,\partial,\mu) \xrightarrow{\cong} (W,\widetilde{\partial},\widetilde{\mu})$. Conversely, given another element $(\partial',\mu') \in \mathrm{Mod}^U(W)$, if there is a DG U-module isomorphism $\beta\colon (W,\partial,\mu) \xrightarrow{\cong} (W,\partial',\mu')$, then $\beta \in \mathrm{GL}(W)_0$ and $(\partial',\mu') = \beta\cdot(\partial,\mu)$. In other words, the orbits in $\mathrm{Mod}^U(W)$ under the action of $\mathrm{GL}(W)_0$ are the isomorphism classes of DG U-module structures on W. Given an element $M = (\partial,\mu) \in \mathrm{Mod}^U(W)$, the orbit $\mathrm{GL}(W)_0 \cdot M$ is locally closed in $\mathrm{Mod}^U(W)$; see [18, Chapter II, §5.3].

Note that the maps defining the action of $\mathrm{GL}(W)_0$ on $\mathrm{Mod}^U(W)$ are regular, that is, determined by polynomial functions. This is because the inversion map $\alpha \mapsto \alpha^{-1}$ on $\mathrm{GL}(W)_0$ is regular, as is the multiplication of matrices corresponding to the compositions defining $\widetilde{\partial}$ and $\widetilde{\mu}$.

Next, we consider even more geometry by identifying tangent spaces to two of our objects of study.

Notation 8.12 We work in the setting of Notation 8.1. Let $F[\varepsilon] := F\varepsilon \bigoplus F$ be the algebra of dual numbers, where $\varepsilon^2 = 0$ and $|\varepsilon| = 0$. For our convenience, we write elements of $F[\varepsilon]$ as column vectors: $a\varepsilon + b = \left[\begin{smallmatrix} a \\ b \end{smallmatrix}\right]$. We identify $U[\varepsilon] := F[\varepsilon] \otimes_F U$ with $U\varepsilon \bigoplus U \cong U \bigoplus U$, and $W[\varepsilon] := F[\varepsilon] \otimes_F W$ with $W\varepsilon \bigoplus W \cong W \bigoplus W$. Using this protocol, we have $\partial_i^{U[\varepsilon]} = \left[\begin{smallmatrix} \partial_i^U & 0 \\ 0 & \partial_i^U \end{smallmatrix}\right]$.

Let $\mathrm{Mod}^{U[\varepsilon]}(W[\varepsilon])$ denote the set of all ordered pairs (∂,μ) making $W[\varepsilon]$ into a DG $U[\varepsilon]$-module. Let $\mathrm{End}_{F[\varepsilon]}(W[\varepsilon])_0$ denote the set of $F[\varepsilon]$-linear endomorphisms of $W[\varepsilon]$ that are homogeneous of degree 0. Let $\mathrm{GL}(W[\varepsilon])_0$ denote the set of $F[\varepsilon]$-linear automorphisms of $W[\varepsilon]$ that are homogeneous of degree 0, that is, the invertible elements of $\mathrm{End}_{F[\varepsilon]}(W[\varepsilon])_0$.

Given an element $M = (\partial,\mu) \in \mathrm{Mod}^U(W)$, the tangent space $\mathsf{T}_M^{\mathrm{Mod}^U(W)}$ is the set of all ordered pairs $(\overline{\partial},\overline{\mu}) \in \mathrm{Mod}^{U[\varepsilon]}(W[\varepsilon])$ that give rise to M modulo ε. The tangent space $\mathsf{T}_{\mathrm{id}_W}^{\mathrm{GL}(W)_0}$ is the set of all elements of $\mathrm{GL}(W[\varepsilon])_0$ that give rise to id_W modulo ε.

Remark 8.13 Alternate descriptions of the tangent spaces from Notation 8.12 are contained in [32, Lemmas 4.8 and 4.10]. Because of smoothness conditions, the map $\mathrm{GL}(W)_0 \xrightarrow{\cdot M} \mathrm{Mod}^U(W)$ induces a linear transformation $\mathsf{T}_{\mathrm{id}_W}^{\mathrm{GL}(W)_0} \to \mathsf{T}_M^{\mathrm{Mod}^U(W)}$ whose image is $\mathsf{T}_M^{\mathrm{GL}(W)_0 \cdot M}$; see [32, 4.11. Proof of Theorem B].

The next two results show some profound connections between the algebra and the geometry of the objects under consideration. The ideas behind these results are due to Voigt [42] and Gabriel [22, 1.2 Corollary].

Theorem 8.14 ([32, 4.11. Proof of Theorem B]) *We work in the setting of Notation 8.1. Given an element $M = (\partial, \mu) \in \mathrm{Mod}^U(W)$, there is an isomorphism of abelian groups*
$$T_M^{\mathrm{Mod}^U(W)} \,/\, T_M^{\mathrm{GL}(W)_0 \cdot M} \cong \mathrm{YExt}^1_U(M,M).$$

Proof (Sketch of proof). Using Notation 8.12, let $N = (\overline{\partial}, \overline{\mu})$ be an element of $\mathsf{T}_M^{\mathrm{Mod}^U(W)}$. Since N is a DG $U[\varepsilon]$-module, restriction of scalars along the natural inclusion $U \to U[\varepsilon]$ makes N a DG U-module.

Define $\rho\colon M \to N$ and $\pi\colon N \to M$ by the formulas $\rho(w) := \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$ and $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) := w$. With [32, Lemmas 4.8 and 4.10], one shows that ρ and π are chain maps and that ρ and π are U-linear. In other words, we have an exact sequence
$$0 \to M \xrightarrow{\rho} N \xrightarrow{\pi} M \to 0$$
of DG U-module morphisms. So, we obtain a map $\tau\colon \mathsf{T}_M^{\mathrm{Mod}^U(W)} \to \mathrm{YExt}^1_U(M,M)$ where $\tau(N)$ is the equivalence class of the displayed sequence in $\mathrm{YExt}^1_U(M,M)$. One shows that τ is a surjective abelian group homomorphism with $\mathrm{Ker}(\tau) = \mathsf{T}_M^{\mathrm{GL}(W)_0 \cdot M}$, and the result follows from the First Isomorphism Theorem.

To show that τ is surjective, fix an arbitrary element $\zeta \in \mathrm{YExt}^1_U(M,M)$, represented by the sequence $0 \to M \xrightarrow{f} Z \xrightarrow{g} M \to 0$. In particular, this is an exact sequence of F-complexes, so it is degree-wise split. This implies that we have a commutative diagram of graded vector spaces:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "f"] \arrow[d, "="'] & Z \arrow[r, "g"] \arrow[d, "\vartheta"] & M \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & M \arrow[r, "\rho"] & W[\varepsilon] \arrow[r, "\pi"] & M \arrow[r] & 0
\end{tikzcd}$$

where $\rho(w) = \left[\begin{smallmatrix} w \\ 0 \end{smallmatrix}\right]$, $\pi\left(\left[\begin{smallmatrix} w' \\ w \end{smallmatrix}\right]\right) = w$, and ϑ is an isomorphism of graded F-vector spaces. The map ϑ allows us to endow $W[\varepsilon]$ with a DG $U[\varepsilon]$-module structure $(\overline{\partial}, \overline{\mu})$ that gives rise to M modulo ε. So we have $N := (\overline{\partial}, \overline{\mu}) \in \mathsf{T}_M^{\mathrm{Mod}^U(W)}$. Furthermore, we have $\tau(N) = \zeta$, so τ is surjective.

See [32, 4.11. Proof of Theorem B] for more details. □

Corollary 8.15 ([32, Corollary 4.12]) *We work in the setting of Notation 8.1. Let C be a semidualizing DG U-module, and let $s \geqslant \sup(C)$. Set $M = \tau(C)_{(\leqslant s)}$ and $W = M^\natural$. Then the orbit $\mathrm{GL}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$.*

Proof. Fact 7.37 implies that $\mathrm{YExt}^1_U(M,M) = 0$, so $\mathsf{T}_M^{\mathrm{Mod}^U(W)} = \mathsf{T}_M^{\mathrm{GL}(W)_0 \cdot M}$ by Theorem 8.14. As the orbit $\underline{\mathrm{GL}}(W)_0 \cdot M$ is smooth and locally closed, this implies that $\underline{\mathrm{GL}}(W)_0 \cdot M$ is open in $\mathrm{Mod}^U(W)$. See [32, Corollary 4.12] for more details. □

We are now in a position to state and prove our version of Happel's result [25, proof of first proposition in section 3] that was used in the proof of Theorem 2.13.

Lemma 8.16 ([32, Lemma 5.1]) *We work in the setting of Notation 8.1. The set $\mathfrak{S}_W(U)$ of quasiisomorphism classes of semi-free semidualizing DG U-modules C such that $s \geqslant \sup(C)$, $C_i = 0$ for all $i < 0$, and $(\tau(C)_{(\leqslant s)})^{\natural} \cong W$ is finite.*

Proof. Fix a representative C for each quasiisomorphism class in $\mathfrak{S}_W(U)$, and write $[C] \in \mathfrak{S}_W(U)$ and $M_C = \tau(C)_{(\leqslant s)}$.

Let $[C], [C'] \in \mathfrak{S}_W(U)$. If $\mathrm{GL}(W)_0 \cdot M_C = \mathrm{GL}(W)_0 \cdot M_{C'}$, then $[C] = [C']$: indeed, Discussion 8.8 explains the second step in the next display

$$C \simeq M_C \cong M_{C'} \simeq C'$$

and the remaining steps follow from the assumptions $s \geqslant \sup(C)$ and $s \geqslant \sup(C')$, by Exercise 5.27.

Now, each orbit $\mathrm{GL}(W)_0 \cdot M_C$ is open in $\mathrm{Mod}^U(W)$ by Corollary 8.15. Since $\mathrm{Mod}^U(W)$ is a subset of an affine space over F, it is quasi-compact, so it can only have finitely many open orbits. By the previous paragraph, this implies that there are only finitely many distinct elements $[C] \in \mathfrak{S}_W(U)$. $\square$

We conclude this section with the third and final part of the proof of Theorem 1.4.

8.17 (Final part of the proof of Theorem 1.4) We need to prove that $\mathfrak{S}(U)$ is finite where $U = k \otimes_Q A$. Set $s = \dim(R) - \mathrm{depth}(R) + n$. One uses various accounting principles to prove that every semidualizing DG U-module is equivalent to a semidualizing DG U-module C' such that $\mathrm{H}_i(C') = 0$ for all $i < 0$ and for all $i > s$. Let $L \xrightarrow{\simeq} C'$ be a minimal semi-free resolution of C' over U. The conditions $\sup(L) = \sup(C') \leqslant s$ imply that L (and hence C') is quasiisomorphic to the truncation $\widetilde{L} := \tau(L)_{\leqslant s}$. We set $W := \widetilde{L}^{\natural}$ and work in the setting of Notation 8.1.

One then uses further accounting principles to prove that there is an integer $\lambda \geqslant 0$, depending only on R and U, such that $\sum_{i=0}^{s} r_i \leqslant \lambda$. Compare this with Lemma 2.9. (Recall that r_i and other quantities are fixed in Notation 8.1.) Then, because there are only finitely many $(r_0, \ldots, r_s) \in \mathbb{N}^{s+1}$ with $\sum_{i=0}^{s} r_i \leqslant \lambda$, there are only finitely many W that occur from this construction, say $W^{(1)}, \ldots, W^{(b)}$. Lemma 8.16 implies that $\mathfrak{S}(U) = \mathfrak{S}_{W^{(1)}}(U) \cup \cdots \cup \mathfrak{S}_{W^{(b)}}(U) \cup \{[U]\}$ is finite. $\square$

9 Applications of Semidualizing Modules

This section contains three applications of semidualizing modules, to indicate why Theorem 1.4 might be interesting.

Assumption 9.1 Throughout this section, $(R, \mathfrak{m}, k)$ is local.

Application I. Asymptotic Behavior of Bass Numbers

Our first application shows that the existence of non-trivial semidualizing modules forces the sequence of Bass numbers of a local ring to be unbounded. This partially answers a question of Huneke.

Definition 9.2 The *ith Bass number* of R is $\mu_R^i := \mathrm{rank}_k(\mathrm{Ext}_R^i(k,R))$. The *Bass series* of R is the formal power series $I_R(t) = \sum_{i=1}^{\infty} \mu_R^i t^i$.

Remark 9.3 The Bass numbers of R contain important structural information about the minimal injective resolution J of R. They also keep track of the depth and injective dimension of R:

$$\mathrm{depth}(R) = \min\{i \geqslant 0 \mid \mu_R^i \neq 0\}$$
$$\mathrm{id}_R(R) = \sup\{i \geqslant 0 \mid \mu_R^i \neq 0\}.$$

In particular, R is Gorenstein if and only if the sequence $\{\mu_R^i\}$ is eventually 0. If R has a dualizing module D, then the Bass numbers of R are related to the Betti numbers of D by the formula

$$\mu_R^{i+\mathrm{depth}(R)} = \beta_i^R(D) := \mathrm{rank}_k(\mathrm{Ext}_R^i(D,k)).$$

Viewed in the context of the characterization of Gorenstein rings in Remark 9.3, the next question is natural, even if it is a bit bold.[12]

Question 9.4 (Huneke) If the sequence $\{\mu_R^i\}$ is bounded, must R be Gorenstein? Equivalently, if R is not Gorenstein, must the sequence $\{\mu_R^i\}$ be unbounded?

The connection between semidualizing modules and Huneke's question is found in the following result. It shows that Huneke's question reduces to the case where R has only trivial semidualizing modules. It is worth noting that the same conclusion holds when R is not assumed to be Cohen-Macaulay and C is a semidualizing *DG* module that is neither free nor dualizing. However, since we have not talked about dualizing DG modules, we only state the module case.

Theorem 9.5 ([38, Theorem B]) *If R is Cohen-Macaulay and has a semidualizing module C that is neither free nor dualizing, then the sequence $\{\mu_R^i\}$ is unbounded.*

Proof (Sketch of proof). Pass to the completion of R to assume that R is complete. This does not change the sequence $\{\mu_R^i\}$ nor the assumptions on C. As R is complete, it has a dualizing module D, and one has $\mu_R^i = \mathrm{rank}_k(\mathrm{Ext}_R^{i-d}(D,k))$ for all i, where $d = \mathrm{depth}(R)$. Thus, it suffices to show that the sequence $\{\mathrm{rank}_k(\mathrm{Ext}_R^i(D,k))\}$ is unbounded. With $C' = \mathrm{Hom}_R(C,D)$, one has $\mathrm{rank}_k(\mathrm{Ext}_R^i(C,k)), \mathrm{rank}_k(\mathrm{Ext}_R^i(C',k)) \geqslant 1$ for all $i \geqslant 0$. Hence, the computation

[12] As best we know, Huneke has not posed this question in print.

$$\operatorname{rank}_k(\operatorname{Ext}_R^i(D,k)) = \sum_{p=0}^{i} \operatorname{rank}_k(\operatorname{Ext}_R^p(C,k)) \operatorname{rank}_k(\operatorname{Ext}_R^{p-i}(C',k)) \geqslant \sum_{p=0}^{i} 1 = p+1$$

gives the desired unboundedness. $\square$

Application II. Structure of Quasi-deformations

Our second application shows how semidualizing modules can be used to improve given structures. Specifically, one can use a particular semidualizing module to improve the closed fibre of a given quasi-deformation.

Definition 9.6 ([8, (1.1) and (1.2)]) A *quasi-deformation* of R is a diagram of local ring homomorphisms $R \xrightarrow{\varphi} R' \xleftarrow{\tau} Q$ such that φ is flat and τ is surjective with kernel generated by a Q-regular sequence. A finitely generated R-module M has *finite CI-dimension* if there is a quasideformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M) < \infty$.

Remark 9.7 A straightforward localization and completion argument shows that if M is an R-module of finite CI-dimension, then there is a quasideformation $R \to R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M)$ is finite, Q is complete, and $R'/\mathfrak{m}R'$ is artinian, hence Cohen-Macaulay.

The next result is a souped-up version of the previous remark. In contrast to the application of semidualizing modules in Theorem 9.5, this one does not refer to any semidualizing modules in the statement. Instead, in its proof, one uses a semidualizing module to improve the quasideformation given by definition to one satisfying the desired conclusions.

Theorem 9.8 ([37, Theorem F]) *If M is an R-module of finite CI-dimension, then there is a quasideformation $R \to R' \leftarrow Q$ such that* $\operatorname{pd}_Q(R' \otimes_R M) < \infty$ *and such that $R'/\mathfrak{m}R'$ is artinian and Gorenstein.*

Proof (Sketch of proof). By Remark 9.7, there is a quasideformation $R \xrightarrow{\varphi} R' \leftarrow Q$ such that $\operatorname{pd}_Q(R' \otimes_R M)$ is finite, Q is complete, and $R'/\mathfrak{m}R'$ is artinian. We work to improve this quasi-deformation.

A relative version of Cohen's Structure Theorem due to Avramov, Foxby, and Herzog [7, (1.1) Theorem] provides "Cohen factorization" of φ, that is a commutative diagram of local ring homomorphisms

$$\begin{tikzcd} & R'' \arrow[dr, "\varphi'"] & \\ R \arrow[ur, "\dot{\varphi}"] \arrow[rr, "\varphi"] & & R' \end{tikzcd}$$

such that $\dot{\varphi}$ is flat, $R''/\mathfrak{m}R''$ is regular, and φ' is surjective. Since φ is flat and $R'/\mathfrak{m}R'$ is Cohen-Macaulay, it follows that R' is perfect over R''. From this, we conclude that $\operatorname{Ext}^i_{R''}(R',R'') = 0$ for all $i \neq c$ where $c = \operatorname{depth}(R'') - \operatorname{depth}(R')$, and that

$D^\varphi := \mathrm{Ext}^c_{R''}(R', R'')$ is a semidualizing R'-module. (This is the "relative dualizing module" of Avramov and Foxby [5].) This implies that $\mathrm{Ext}^2_{R'}(D^\varphi, D^\varphi) = 0$, so there is a semidualizing Q-module B such that $D^\varphi \cong R' \otimes_Q B$. (This essentially follows from a lifting theorem of Auslander, Ding, and Solberg [3, Proposition 1.7], since Q is complete.)

The desired quasi-deformation is in the bottom row of the following diagram

$$\begin{tikzcd} & R' \arrow[d] & Q \arrow[l] \arrow[d] \\ R \arrow[ur, "\varphi"] \arrow[r] & R' \ltimes D^\varphi & Q \ltimes B \arrow[l] \end{tikzcd}$$

where $Q \ltimes B$ and $R' \ltimes D^\varphi$ are "trivial extensions", i.e., Nagata's "idealizations". □

Application III. Bass Series of Local Ring Homomorphisms

Our third application of semidualizing modules is a version of Theorem 1.3 from [5] where φ is only assumed to have finite G-dimension, defined in the next few items.

Definition 9.9 A finitely generated R-module G is *totally reflexive* if one has $G \cong \mathrm{Hom}_R(\mathrm{Hom}_R(G,R),R)$ and $\mathrm{Ext}^i_R(G,R) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(G,R),R)$ for all $i \geqslant 1$. A finitely generated R-module M has *finite G-dimension* if there is an exact sequence $0 \to G_n \to \cdots \to G_0 \to M \to 0$ such that each G_i is totally reflexive.

Remark 9.10 If M is an R-module of finite G-dimension, then $\mathrm{Ext}^i_R(M,R) = 0$ for $i \gg 0$ and $\mathrm{Ext}_R^{\mathrm{depth}(R)-\mathrm{depth}_R(M)}(M,R) \neq 0$.

Definition 9.11 Let $R \xrightarrow{\varphi} S$ be a local ring homomorphism, and let $S \xrightarrow{\psi} \widehat{S}$ be the natural map where $\widehat{S}$ is the completion of S. Fix a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of the "semi-completion of φ", i.e., the composition $R \xrightarrow{\psi\varphi} \widehat{S}$ (see the proof of Theorem 9.8). We say that φ has *finite G-dimension* if $\widehat{S}$ has finite G-dimension over R'. Moreover, the map φ is *quasi-Gorenstein* if it has finite G-dimension and $\mathrm{Ext}^i_R(R', \widehat{S}) = 0$ for all $i \neq \mathrm{depth}(R') - \mathrm{depth}(\widehat{S})$.

The next result is the aforementioned improvement of Theorem 1.3. As with Theorem 9.8, note that the statement does not involve semidualizing modules.

Theorem 9.12 ([5, (7.1) Theorem]) *Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a local ring homomorphism of finite G-dimension. Then there is a formal Laurent series $I_\varphi(t)$ with non-negative integer coefficients such that $I_S(t) = I_R(t) I_\varphi(t)$. In particular, if S is Gorenstein, then so is R.*

Proof (Sketch of proof when φ is quasi-Gorenstein). Fix a Cohen factorization $R \xrightarrow{\dot{\varphi}} R' \xrightarrow{\varphi'} \widehat{S}$ of the semi-completion of φ, and set $d = \mathrm{depth}(R') - \mathrm{depth}(\widehat{S})$. Since φ

is quasi-Gorenstein, the $\widehat{S}$-module $D' = \operatorname{Ext}^d_R(R', \widehat{S})$ is semidualizing. (Again, this is Avramov and Foxby's relative dualizing module [5].) If l denotes the residue field of S, then the series $I_\varphi := \sum_i \operatorname{rank}_l(\operatorname{Ext}^{i+d'}_{\widehat{S}}(D', l))t^i$ satisfies the desired conditions where $d' = \operatorname{depth}(R) - \operatorname{depth}(S)$. $\square$

10 Sketches of Solutions to Exercises

10.1 (Sketch of Solution to Exercise 2.8) As S is R-flat, there is an isomorphism

$$\operatorname{Ext}^i_S(S \otimes_R C, S \otimes_R C) \cong S \otimes_R \operatorname{Ext}^i_R(C, C)$$

for each i. It follows that (1) if $\operatorname{Ext}^i_R(C,C) = 0$, then $\operatorname{Ext}^i_S(S \otimes_R C, S \otimes_R C) = 0$, and (2) if φ is faithfully flat and $\operatorname{Ext}^i_S(S \otimes_R C, S \otimes_R C) = 0$, then $\operatorname{Ext}^i_R(C,C) = 0$. Similarly, there is a commutative diagram

$$\begin{tikzcd}
S \arrow[r, "S\otimes_R \chi^R_C"] \arrow[d, "\chi^S_{S\otimes_R C}"'] & S \otimes_R \operatorname{Hom}_R(C,C) \arrow[dl, "\cong"] \\
\operatorname{Hom}_S(S \otimes_R C, S \otimes_R C) &
\end{tikzcd}$$

so (1) if χ^R_C is an isomorphism, then so is $\chi^S_{S\otimes_R C}$, and (2) if φ is faithfully flat and $\chi^S_{S\otimes_R C}$ is an isomorphism, then so is χ^R_C. $\square$

10.2 (Sketch of Solution to Exercise 3.4)

(a) It is routine to show that $\partial_n^{\operatorname{Hom}_R(X,Y)}$ is R-linear and maps $\operatorname{Hom}_R(X,Y)_n$ to $\operatorname{Hom}_R(X,Y)_{n-1}$. To show that $\operatorname{Hom}_R(X,Y)$ is an R-complex, we compute:

$$\begin{aligned}
&\partial_{n-1}^{\operatorname{Hom}_R(X,Y)}(\partial_n^{\operatorname{Hom}_R(X,Y)}(\{f_p\})) \\
&= \partial_{n-1}^{\operatorname{Hom}_R(X,Y)}(\{\partial^Y_{p+n} f_p - (-1)^n f_{p-1}\partial^X_p\}) \\
&= \{\partial^Y_{p+n-1}[\partial^Y_{p+n} f_p - (-1)^n f_{p-1}\partial^X_p] - (-1)^{n-1}[\partial^Y_{p+n-1} f_{p-1} - (-1)^n f_{p-2}\partial^X_{p-1}]\partial^X_p\} \\
&= \{\underbrace{\partial^Y_{p+n-1}\partial^Y_{p+n}}_{=0} f_p \underbrace{-(-1)^n\partial^Y_{p+n} f_{p-1}\partial^X_p - (-1)^{n-1}\partial^Y_{p+n-1} f_{p-1}\partial^X_p}_{=0} + f_{p-2}\underbrace{\partial^X_{p-1}\partial^X_p}_{=0}\} \\
&= 0.
\end{aligned}$$

(b) For $f = \{f_p\} \in \operatorname{Hom}_R(X,Y)_0$, we have

$$\partial_0^{\operatorname{Hom}_R(X,Y)}(\{f_p\}) = \{\partial^Y_p f_p - f_{p-1}\partial^X_p\}.$$

Hence, f is a chain map if and only if $\partial_p^Y f_p - f_{p-1}\partial_p^X = 0$ for all p, which is equivalent to the commutativity of the given diagram.

(c) This follows by the fact $\partial_0^{\mathrm{Hom}_R(X,Y)} \circ \partial_1^{\mathrm{Hom}_R(X,Y)} = 0$, from part (a).

(d) Since $\partial_1^{\mathrm{Hom}_R(X,Y)}(\{s_p\}) = \{\partial_{p+1}^Y s_p + s_{p-1}\partial_p^X\}$, this follows by definition. □

10.3 (Sketch of Solution to Exercise 3.5) Let $\tau\colon \mathrm{Hom}_R(R,X) \to X$ be given by $\tau_n(\{f_p\}) = f_n(1)$. We consider R as a complex concentrated in degree 0, so for all $p \neq 0$ we have $R_p = 0$, and for all n we have $\partial_n^R = 0$. It follows that, for all n and all $f = \{f_p\} \in \mathrm{Hom}_R(R,X)_n$ and all $p \neq 0$, we have $f_p = 0$. Thus, for each n the natural maps $\mathrm{Hom}_R(R,X)_n \xrightarrow{\cong} \mathrm{Hom}_R(R,X_n) \xrightarrow{\cong} X_n$ are R-module isomorphisms, the composition of which is τ_n. To show that τ is a chain map, we compute:

$$\begin{aligned}
\tau_{n-1}(\partial_n^{\mathrm{Hom}_R(R,X)}(\{f_p\})) &= \tau_{n-1}(\partial_n^{\mathrm{Hom}_R(R,X)}(\ldots,0,f_0,0,\ldots)) \\
&= \tau_{n-1}(\ldots,0,\partial_n^X f_0, -(-1)^n f_0\partial_1^R, 0,\ldots) \\
&= \tau_{n-1}(\ldots,0,\partial_n^X f_0,0,0,\ldots) \\
&= \partial_n^X(f_0(1)) \\
&= \partial_n^X(\tau_n(\{f_p\})).
\end{aligned}$$

(Note that these steps are optimal for presentation, in some sense, but they do not exactly represent the thought process we used to find the solution. Instead, we computed and simplified $\tau_{n-1}(\partial_n^{\mathrm{Hom}_R(R,X)}(\{f_p\}))$ and $\partial_n^X(\tau_n(\{f_p\}))$ separately and checked that the resulting expression was the same for both. Similar comments apply to many solutions.) □

10.4 (Sketch of Solution to Exercise 3.6) Let X be an R-complex, and let M be an R-module.

(a) Write X_* for the complex

$$X_* = \cdots \xrightarrow{(\partial_{n+1}^X)_*} (X_n)_* \xrightarrow{(\partial_n^X)_*} (X_{n-1})_* \xrightarrow{(\partial_{n-1}^X)_*} \cdots.$$

We consider M as a complex concentrated in degree 0, so for all $p \neq 0$ we have $M_p = 0$, and for all n we have $\partial_n^M = 0$. It follows that, for all n and all $f = \{f_p\} \in \mathrm{Hom}_R(M,X)_n$ and all $p \neq 0$, we have $f_p = 0$. Thus, for each n the natural map $\tau_n\colon \mathrm{Hom}_R(M,X)_n \xrightarrow{\cong} \mathrm{Hom}_R(M,X_n) = (X_*)_n$ is an R-module isomorphism. Thus, it remains to show that the map $\tau\colon \mathrm{Hom}_R(M,X) \to X_*$ given by $f = \{f_p\} \mapsto f_0$ is a chain map. We compute:

$$\begin{aligned}
\tau_{n-1}(\partial_n^{\mathrm{Hom}_R(M,X)}(\{f_p\})) &= \tau_{n-1}(\partial_n^{\mathrm{Hom}_R(M,X)}(\ldots,0,f_0,0,\ldots)) \\
&= \tau_{n-1}(\ldots,0,\partial_n^X f_0, -(-1)^n f_0\partial_1^M, 0,\ldots) \\
&= \tau_{n-1}(\ldots,0,\partial_n^X f_0,0,0,\ldots) \\
&= \partial_n^X f_0.
\end{aligned}$$

This yields

$$\tau_{n-1}(\partial_n^{\mathrm{Hom}_R(M,X)}(\{f_p\})) = \partial_n^X f_0 = (\partial_n^X)_*(f_0) = \partial_n^{X_*}(f_0) = \partial_n^{X_*}(\tau_n(\{f_p\}))$$

as desired

(b) Write X^* and $X^\dagger$ for the complexes

$$X^* = \cdots \xrightarrow{(-1)^n(\partial_n^X)^*} X_n^* \xrightarrow{(-1)^{n+1}(\partial_{n+1}^X)^*} X_{n+1}^* \xrightarrow{(-1)^{n+2}(\partial_{n+2}^X)^*} \cdots$$

$$X^\dagger = \cdots \xrightarrow{(\partial_n^X)^*} X_n^* \xrightarrow{(\partial_{n+1}^X)^*} X_{n+1}^* \xrightarrow{(\partial_{n+2}^X)^*} \cdots.$$

Note that the displayed pieces for X^* are in degree $-n$, $1-n$, and similarly for $X^\dagger$. We prove that $\mathrm{Hom}_R(X,M) \cong X^* \cong X^\dagger$.

As in part (a), for all $p \neq 0$ we have $M_p = 0$, and for all n we have $\partial_n^M = 0$. It follows that, for all n and all $f = \{f_p\} \in \mathrm{Hom}_R(X,M)_n$ and all $p \neq -n$, we have $f_p = 0$. Thus, for each n the map $\tau_n\colon \mathrm{Hom}_R(X,M)_n \xrightarrow{\cong} \mathrm{Hom}_R(X_{-n},M) = (X^*)_n$ given by $\{f_p\} \mapsto f_{-n}$ is an R-module isomorphism. Thus, for the isomorphism $\mathrm{Hom}_R(M,X) \cong X^*$, it remains to show that $\tau\colon \mathrm{Hom}_R(X,M) \to X^*$ is a chain map. We compute:

$$\begin{aligned}
\tau_{n-1}(\partial_n^{\mathrm{Hom}_R(X,M)}(\{f_p\})) &= \tau_{n-1}(\partial_n^{\mathrm{Hom}_R(X,M)}(\ldots,0,f_{-n},0,\ldots)) \\
&= \tau_{n-1}(\ldots,0,\partial_0^M f_{-n}, -(-1)^n f_{-n}\partial_{1-n}^X, 0,\ldots) \\
&= \tau_{n-1}(\ldots,0,(-1)^{n-1} f_{-n}\partial_{1-n}^X, 0,0,\ldots) \\
&= (-1)^{n-1} f_{-n}\partial_{1-n}^X \\
&= (-1)^{1-n}(\partial_{1-n}^X)^*(f_{-n}) \\
&= \partial_n^{X^*}(f_{-n}) \\
&= \partial_n^{X^*}(\tau_n(\{f_p\})).
\end{aligned}$$

For the isomorphism $X^* \cong X^\dagger$, we first observe that $X^\dagger$ is an R-complex. Next, note the following: given an R-complex Y, the following diagram describes an isomorphism of R-complexes.

$$\begin{tikzcd}
Y_{4n+2} \arrow[r, "(-1)^{-4n-1}\partial^Y_{4n+2}"] \arrow[d, "(-1)^{4n+2}"] & Y_{4n+1} \arrow[r, "(-1)^{-4n}\partial^Y_{4n+1}"] \arrow[d, "(-1)^{4n+1}"'] & Y_{4n} \arrow[r, "(-1)^{-4n+1}\partial^Y_{4n}"] \arrow[d, "(-1)^{4n+1}"'] & Y_{4n-1} \arrow[r, "(-1)^{-4n+2}\partial^Y_{4n-1}"] \arrow[d, "(-1)^{4n}"'] & Y_{4n-2} \arrow[d, "(-1)^{4n-2}"'] \\
Y_{4n+2} \arrow[r, "\partial^Y_{4n+2}"'] & Y_{4n+1} \arrow[r, "\partial^Y_{4n+1}"'] & Y_{4n} \arrow[r, "\partial^Y_{4n}"'] & Y_{4n-1} \arrow[r, "\partial^Y_{4n-1}"'] & Y_{4n-2}
\end{tikzcd}$$

Now, apply this observation to $X^\dagger$. $\square$

10.5 (Sketch of Solution to Exercise 3.7) (a) Let $f\colon X \to Y$ be a chain map. By definition, we have $f_i\partial_{i+1}^X = \partial_{i+1}^Y f_{i+1}$. It follows readily that $f_i(\mathrm{Im}(\partial_{i+1}^X)) \subseteq \mathrm{Im}(\partial_{i+1}^Y)$

and $f_i(\mathrm{Ker}(\partial_i^X)) \subseteq \mathrm{Ker}(\partial_i^Y)$. From this, it is straightforward to show that the map $\mathrm{Ker}(\partial_i^X)/\mathrm{Im}(\partial_{i+1}^X) \to \mathrm{Ker}(\partial_i^Y)/\mathrm{Im}(\partial_{i+1}^Y)$ given by $\overline{x} \mapsto \overline{f_i(x)}$ is a well-defined R-module homomorphism, as desired.

(b) Assume now that f is null-homotopic. By definition, there is an element $s = \{s_p\} \in \mathrm{Hom}_R(X,Y)_1$ such that $\{f_p\} = \partial_1^{\mathrm{Hom}_R(X,Y)}(\{s_p\}) = \{\partial_{p+1}^Y s_p + s_{p-1}\partial_p^X\}$. It follows that for each i and each $\overline{x} \in \mathrm{H}_i(X)$, one has

$$\mathrm{H}_i(f)(\overline{x}) = \overline{f_i(x)} = \overline{\underbrace{\partial_{i+1}^Y(s_i(x))}_{\in \mathrm{Im}(\partial_{i+1}^Y)} + s_{i-1}(\underbrace{\partial_i^X(x)}_{=0})} = 0$$

in $\mathrm{H}_i(Y)$. $\square$

10.6 (Sketch of Solution to Exercise 3.9) Let $f\colon X \xrightarrow{\cong} Y$ be an isomorphism between the R-complexes X and Y. Then for each i the map f_i induces isomorphisms $\mathrm{Ker}(\partial_i^X) \xrightarrow{\cong} \mathrm{Ker}(\partial_i^Y)$ and $\mathrm{Im}(\partial_{i+1}^X) \xrightarrow{\cong} \mathrm{Im}(\partial_{i+1}^Y)$. It follows that f_i induces an isomorphism $\mathrm{Ker}(\partial_i^X)/\mathrm{Im}(\partial_{i+1}^X) \xrightarrow{\cong} \mathrm{Ker}(\partial_i^Y)/\mathrm{Im}(\partial_{i+1}^Y)$, as desired. $\square$

10.7 (Sketch of Solution to Exercise 3.10) Let M be an R-module with augmented projective resolution P^+.

$$P^+ = \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\tau} M \to 0.$$

It is straightforward to check that the following diagram commutes

$$\begin{array}{ccc} P & & \cdots \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \longrightarrow 0 \\ {\scriptstyle t}\downarrow & & \quad\quad\quad\; \downarrow \quad\;\; {\scriptstyle \tau}\downarrow \quad\quad \downarrow \\ M & & \quad\quad\quad\; 0 \longrightarrow M \longrightarrow 0. \end{array}$$

The exactness of P^+ and the definition of P implies that $\mathrm{H}_i(P) = 0 = \mathrm{H}_i(M)$ for $i \neq 0$. Thus, to show that t is a quasiisomorphism, it suffices to show that $\mathrm{H}_0(t)\colon \mathrm{H}_0(P) \to \mathrm{H}_0(M)$ is an isomorphism. Notice that this can be identified with the map $\tau'\colon \mathrm{Coker}(\partial_1^P) \to M$ induced by τ. Since τ is surjective, it is straightforward to show that τ' is surjective. Using the fact that $\mathrm{Ker}(\tau) = \mathrm{Im}(\partial_1^P)$, one shows readily that τ' is injective, as desired.

The case of an injective resolution result is handled similarly. $\square$

10.8 (Sketch of Solution to Exercise 3.14) Fix a chain map $f\colon X \to Y$ and an R-complex Z. Consider a sequence $\{g_p\} \in \mathrm{Hom}_R(Y,Z)_n$. Note that the sequence $\mathrm{Hom}_R(f,Z)_n(\{g_p\}) = \{g_p f_p\}$ is in $\mathrm{Hom}_R(X,Z)_n$, so the map $\mathrm{Hom}_R(f,Z)$ is well-defined and of degree 0. Also, it is straightforward to show that $\mathrm{Hom}_R(f,Z)$ is R-linear. To verify that $\mathrm{Hom}_R(f,Z)$ is a chain map, we compute:

$$\begin{aligned}
\partial_n^{\mathrm{Hom}_R(X,Z)}(\mathrm{Hom}_R(f,Z)_n(\{g_p\})) &= \partial_n^{\mathrm{Hom}_R(X,Z)}(\{g_p f_p\}) \\
&= \{\partial_{p+n}^Z g_p f_p - (-1)^n g_{p-1} f_{p-1} \partial_p^X\} \\
&= \{\partial_{p+n}^Z g_p f_p - (-1)^n g_{p-1} \partial_p^Y f_p\} \\
&= \mathrm{Hom}_R(f,Z)_n(\{\partial_{p+n}^Z g_p - (-1)^n g_{p-1} \partial_p^Y\}) \\
&= \mathrm{Hom}_R(f,Z)_n(\partial_n^{\mathrm{Hom}_R(Y,Z)}(\{g_p\})).
\end{aligned}$$

Note that the third equality follows from the fact that f is a chain map.

The computation for $\mathrm{Hom}_R(Z,f)\colon \mathrm{Hom}_R(Z,X) \to \mathrm{Hom}_R(Z,Y)$ is similar. $\square$

10.9 (Sketch of Solution to Exercise 3.16) It is straightforward to show that $\mu^{X,r}$ is R-linear. (Note that this uses the fact that R is commutative.) To show that $\mu^{X,r}$ is a chain map, we compute:

$$\partial_i^X(\mu_i^{X,r}(x)) = \partial_i^X(rx) = r\partial_i^X(x) = \mu_{i-1}^{X,r}(\partial_i^X(x)).$$

For the induced map $\mathrm{H}_i(\mu^{X,r})$, we have $\mathrm{H}_i(\mu^{X,r})(\overline{x}) = \overline{rx} = r\overline{x}$, as desired. $\square$

10.10 (Sketch of Solution to Exercise 3.17) Exercise 3.16 shows that, for all $r \in R$, the sequence $\chi_0^X(r) = \mu^{X,r}$ is a chain map. That is, $\chi_0^X(r)$ is a cycle in $\mathrm{Hom}_R(X,X)_0$. It follows that the next diagram commutes

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & R \arrow[r] \arrow[d, "\chi_0^X"'] & 0 \arrow[d] \\
\mathrm{Hom}_R(X,X)_1 \arrow[r, "\partial_1^{\mathrm{Hom}_R(X,X)}"] & \mathrm{Hom}_R(X,X)_0 \arrow[r, "\partial_0^{\mathrm{Hom}_R(X,X)}"] & \mathrm{Hom}_R(X,X)_{-1}
\end{tikzcd}$$

so χ^X is a chain map. $\square$

10.11 (Sketch of Solution to Exercise 4.2) Let X, Y, and Z be R-complexes.

(a) It is routine to show that $\partial_n^{X\otimes_R Y}$ is R-linear and maps from $(X\otimes_R Y)_n$ to $(X\otimes_R Y)_{n-1}$. To show that $X\otimes_R Y$ is an R-complex, we compute:

$$\begin{aligned}
&\partial_{n-1}^{X\otimes_R Y}(\partial_n^{X\otimes_R Y}(\ldots,0,x_p\otimes y_{n-p},0,\ldots)) \\
&\quad = \partial_{n-1}^{X\otimes_R Y}(\ldots,0,\partial_p^X(x_p)\otimes y_{n-p},(-1)^p x_p\otimes \partial_{n-p}^Y(y_{n-p}),0,\ldots) \\
&\quad = \partial_{n-1}^{X\otimes_R Y}(\ldots,0,\partial_p^X(x_p)\otimes y_{n-p},0,0,\ldots) \\
&\qquad + \partial_{n-1}^{X\otimes_R Y}(\ldots,0,0,(-1)^p x_p\otimes \partial_{n-p}^Y(y_{n-p}),0,\ldots) \\
&\quad = (\ldots,0,\underbrace{\partial_{p-1}^X(\partial_p^X(x_p))}_{=0}\otimes y_{n-p},(-1)^{p-1}\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p}),0,\ldots) \\
&\qquad + (\ldots,0,(-1)^p\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p}),(-1)^p x_p\otimes\underbrace{\partial_{n-p-1}^Y(\partial_{n-p}^Y(y_{n-p}))}_{=0},0,\ldots).
\end{aligned}$$

The only possibly non-trivial entry in this sum is

$$
\begin{aligned}
&(-1)^{p-1}\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p})+(-1)^p\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p})\\
&\qquad=(-1)^{p-1}[\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p})-\partial_p^X(x_p)\otimes\partial_{n-p}^Y(y_{n-p})]=0
\end{aligned}
$$

so we have $\partial_{n-1}^{X\otimes_R Y}\partial_n^{X\otimes_R Y}=0$.

(b) The isomorphism $X\xrightarrow{\cong}R\otimes_R X$ is given by $x\mapsto(\ldots,0,1\otimes x,0,\ldots)$. Check that this is an isomorphism as in 10.3, using that R is concentrated in degree 0.

(c) As the hint suggests, the isomorphism $g\colon X\otimes_R Y\xrightarrow{\cong}Y\otimes_R X$ requires a sign-change: we define

$$g_n(\ldots,0,x_p\otimes y_{n-p},0,\ldots):=(\ldots,0,(-1)^{p(n-p)}y_{n-p}\otimes x_p,0,\ldots).$$

It is routine to show that $g=\{g_n\}$ is R-linear and maps $(X\otimes_R Y)_n$ to $(Y\otimes_R X)_n$. The following computation shows that g is a chain map:

$$
\begin{aligned}
&g_{n-1}(\partial_n^{X\otimes_R Y}(\ldots,0,x_p\otimes y_{n-p},0,\ldots))\\
&\quad=g_{n-1}(\ldots,0,\partial_p^X(x_p)\otimes y_{n-p},(-1)^p x_p\otimes\partial_{n-p}^Y(y_{n-p}),0,\ldots)\\
&\quad=g_{n-1}(\ldots,0,\partial_p^X(x_p)\otimes y_{n-p},0,\ldots)\\
&\qquad+g_{n-1}(\ldots,0,(-1)^p x_p\otimes\partial_{n-p}^Y(y_{n-p}),0,\ldots)\\
&\quad=(\ldots,0,(-1)^{(p-1)(n-p)}y_{n-p}\otimes\partial_p^X(x_p),0,\ldots)\\
&\qquad+(\ldots,0,(-1)^{p+p(n-p-1)}\partial_{n-p}^Y(y_{n-p})\otimes x_p,0,\ldots)\\
&\quad=(\ldots,0,(-1)^{(p+1)(n-p)}y_{n-p}\otimes\partial_p^X(x_p),0,\ldots)\\
&\qquad+(\ldots,0,(-1)^{p(n-p)}\partial_{n-p}^Y(y_{n-p})\otimes x_p,0,\ldots)\\
&\quad=(\ldots,0,(-1)^{p(n-p)}\partial_{n-p}^Y(y_{n-p})\otimes x_p,(-1)^{(p+1)(n-p)}y_{n-p}\otimes\partial_p^X(x_p),0,\ldots)\\
&\quad=(\ldots,0,(-1)^{p(n-p)}\partial_{n-p}^Y(y_{n-p})\otimes x_p,(-1)^{p(n-p)+(n-p)}y_{n-p}\otimes\partial_p^X(x_p),0,\ldots)\\
&\quad=\partial_n^{Y\otimes_R X}(\ldots,0,(-1)^{p(n-p)}y_{n-p}\otimes x_p,0,\ldots)\\
&\quad=\partial_n^{Y\otimes_R X}(g_n(\ldots,0,x_p\otimes y_{n-p},0,\ldots))
\end{aligned}
$$

Similarly, one shows that the map $h\colon Y\otimes_R X\xrightarrow{\cong}X\otimes_R Y$ defined as

$$h_n(\ldots,0,y_p\otimes x_{n-p},0,\ldots):=(\ldots,0,(-1)^{p(n-p)}x_{n-p}\otimes y_p,0,\ldots)$$

is a chain map. Moreover, one has

$$
\begin{aligned}
g_n(h_n(\ldots,0,y_p\otimes x_{n-p},0,\ldots))&=g_n(\ldots,0,(-1)^{p(n-p)}x_{n-p}\otimes y_p,0,\ldots)\\
&=(\ldots,0,(-1)^{p(n-p)}(-1)^{(n-p)p}y_p\otimes x_{n-p},0,\ldots)\\
&=(\ldots,0,((-1)^{p(n-p)})^2 y_p\otimes x_{n-p},0,\ldots)\\
&=(\ldots,0,y_p\otimes x_{n-p},0,\ldots)
\end{aligned}
$$

so hg is the identity on $Y \otimes_R X$. Similarly, gh is the identity on $X \otimes_R Y$. It follows that h is a two-sided inverse for g, so g is an isomorphism.

(d) For the isomorphism $X \otimes_R (Y \otimes_R Z) \to (X \otimes_R Y) \otimes_R Z$, we change notation. The point is that elements of $X \otimes_R (Y \otimes_R Z)$ are sequences where each entry is itself a sequence. For instance, a generator in $X \otimes_R (Y \otimes_R Z)_n$ is of the form

$$\big(\ldots,(\ldots,0,0,\ldots),(\ldots,0,x_p \otimes (y_q \otimes z_{n-p-q}),0,\ldots),(\ldots,0,0,\ldots),\ldots\big)$$

which we simply write as $x_p \otimes (y_q \otimes z_{n-p-q})$. One has to be careful here not to combine elements illegally: the elements $x_a \otimes (y_b \otimes z_{n-a-b})$ and $x_p \otimes (y_q \otimes z_{n-p-q})$ are only in the same summand of $X \otimes_R (Y \otimes_R Z)_n$ if $a = p$ and $b = q$.

Using this protocol, define $f\colon X \otimes_R (Y \otimes_R Z) \to (X \otimes_R Y) \otimes_R Z$ as

$$f_{p+q+r}(x_p \otimes (y_q \otimes z_r)) = (x_p \otimes y_q) \otimes z_r.$$

(This has no sign-change since no factors are commuted.) As in the previous case, showing that f is an isomorphism reduces to showing that it is a chain map:

$$\begin{aligned}
&\partial^{(X\otimes_R Y)\otimes_R Z}_{p+q+r}(f_{p+q+r}(x_p \otimes (y_q \otimes z_r)))\\
&\quad = \partial^{(X\otimes_R Y)\otimes_R Z}_{p+q+r}((x_p \otimes y_q) \otimes z_r)\\
&\quad = \partial^{X\otimes_R Y}_{p+q}(x_p \otimes y_q) \otimes z_r + (-1)^{p+q}(x_p \otimes y_q) \otimes \partial^Z_r(z_r)\\
&\quad = (\partial^X_p(x_p) \otimes y_q) \otimes z_r + (-1)^p(x_p \otimes \partial^Y_q(y_q)) \otimes z_r + (-1)^{p+q}(x_p \otimes y_q) \otimes \partial^Z_r(z_r)\\
&\quad = f_{p+q+r-1}(\partial^X_p(x_p) \otimes (y_q \otimes z_r)) + (-1)^p x_p \otimes (\partial^Y_q(y_q) \otimes z_r)\\
&\qquad\qquad + (-1)^{p+q} x_p \otimes (y_q \otimes \partial^Z_r(z_r))\\
&\quad = f_{p+q+r-1}(\partial^{X\otimes_R(Y\otimes_R Z)}_{p+q+r}(x_p \otimes (y_q \otimes z_r))).
\end{aligned}$$

The last equality in the sequence follows like the first two. □

10.12 (Sketch of Solution to Exercise 4.8) Fix a chain map $f\colon X \to Y$ and an R-complex Z. For each element $z_p \otimes x_q \in (Z \otimes_R X)_n$, the output $(Z \otimes_R f)_n(z_p \otimes x_q) = z_p \otimes f_q(x_q)$ is in $(X \otimes_R Z)_n$, so the map $Z \otimes_R f\colon Z \otimes_R X \to Z \otimes_R Y$ is well-defined and of degree 0. Also, it is straightforward to show that $Z \otimes_R f$ is R-linear. To show that $Z \otimes_R f$ is a chain map, we compute:

$$\begin{aligned}
\partial^{Z\otimes_R Y}_{p+q}((Z \otimes_R f)_{p+q}(z_p \otimes x_q)) &= \partial^{Z\otimes_R Y}_{p+q}(z_p \otimes f_q(x_q))\\
&= \partial^Z_p(z_p) \otimes f_q(x_q) + (-1)^p z_p \otimes \partial^Y_q(f_q(x_q))\\
&= \partial^Z_p(z_p) \otimes f_q(x_q) + (-1)^p z_p \otimes f_{q-1}(\partial^X_q(x_q))\\
&= (Z \otimes_R f)_{p+q-1}(\partial^Z_p(z_p) \otimes x_q + (-1)^p z_p \otimes \partial^X_q(x_q))\\
&= (Z \otimes_R f)_{p+q-1}(\partial^{Z\otimes_R X}_{p+q}(z_p \otimes x_q)).
\end{aligned}$$

The proof for $f \otimes_R Z$ is similar. □

10.13 (Sketch of Solution to Exercise 4.11) We briefly describe our linear algebra protocols before sketching this solution. For each $n \in \mathbb{N}$, the module R^n consists of the column vectors of size n with entries from R. Under this convention, the R-module homomorphisms $R^m \xrightarrow{f} R^n$ are in bijection with the $n \times m$ matrices with entries from R. Moreover, the matrix representing f has columns $f(e_1), \ldots, f(e_m)$ where $e_1, \ldots, e_m$ is the standard basis for R^m.

More generally, given free R-modules V and W with ordered bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ respectively, the R-module homomorphisms $V \xrightarrow{f} W$ are in bijection with the $n \times m$ matrices with entries from R. Moreover, the jth column of the matrix representing f with these bases is $\begin{pmatrix} a_{1,j} \\ a_{2,j} \\ \vdots \\ a_{n,j} \end{pmatrix}$ where $f(v_j) = \sum_{i=1}^n a_{i,j} w_i$.

Note that these protocols allow for function composition to be represented by matrix multiplication in the same order: if f and g are represented by matrices A and B, respectively, then gf is represented by BA, using the same ordered bases.

Now for the sketch of the solution. By definition, we have

$$\begin{aligned} K^R(x) = &\quad 0 \to Re \xrightarrow{x} R1 \to 0 \\ K^R(y) = &\quad 0 \to Rf \xrightarrow{y} R1 \to 0 \\ K^R(z) = &\quad 0 \to Rg \xrightarrow{z} R1 \to 0. \end{aligned}$$

The notation indicates that, in each case, the basis vector in degree 0 is 1 and the basis vectors in degree 1 are e, f, and g, respectively.

We have $K^R(x,y) = K^R(x) \otimes_R K^R(y)$, by definition. It follows that $K^R(x,y)_n = \bigoplus_{i+j=n} K^R(x)_i \otimes K^R(y)_j$. Since $K^R(x)_i = 0 = K^R(y)_i$ for all $i \neq 0,1$, it follows readily that $K^R(x,y)_n = 0$ for all $n \neq 0,1,2$. In degree 0, we have

$$K^R(x,y)_0 = K^R(x)_0 \otimes_R K^R(y)_0 = R1 \otimes_R R1 \cong R$$

with basis vector $1 \otimes 1$. Similarly, we have

$$K^R(x,y)_1 = (Re \otimes_R R1) \bigoplus (R1 \otimes_R Rf) \cong R^2$$

with ordered basis $e \otimes 1$, $1 \otimes f$. In degree 2, we have

$$K^R(x,y)_2 = Re \otimes_R Rf \cong R$$

with basis vector $e \otimes f$. In particular, $K^R(x,y)$ has the following shape:

$$K^R(x,y) = \quad 0 \to R \xrightarrow{d_2} R^2 \xrightarrow{d_1} R \to 0.$$

Using the linear algebra protocols described above, we identify an element $re \otimes 1 + s1 \otimes f \in K^R(x,y)_1 \cong R^2$ with the column vector $\binom{r}{s}$. It follows that d_2 is a 2×1 matrix, and d_1 is a 1×2 matrix, which we identify from the images of the corresponding basis vectors. First, for d_2:

$$d_2(e\otimes f)=\partial_1^{K^R(x)}(e)\otimes f+(-1)^{|e|}e\otimes\partial_1^{K^R(y)}(f)=x1\otimes f-ye\otimes 1.$$

This corresponds to the column vector $\left(\begin{smallmatrix}-y\\x\end{smallmatrix}\right)$, and it follows that d_2 is represented by the matrix $\left(\begin{smallmatrix}-y\\x\end{smallmatrix}\right)$. For d_1, we have two basis vectors to consider, in order:

$$d_1(e\otimes 1)=\partial_1^{K^R(x)}(e)\otimes 1+(-1)^{|e|}e\otimes\partial_1^{K^R(y)}(1)=x1\otimes 1$$
$$d_1(1\otimes f)=\partial_1^{K^R(x)}(1)\otimes f+(-1)^{|1|}1\otimes\partial_1^{K^R(y)}(f)=y1\otimes 1.$$

It follows that the map d_1 is represented by the row matrix $(x\ y)$, so in summary:

$$K^R(x,y)=\quad 0\to R\xrightarrow{\left(\begin{smallmatrix}-y\\x\end{smallmatrix}\right)}R^2\xrightarrow{(x\ y)}R\to 0.$$

The condition $d_1d_2=0$ follows from the fact that $xy=yx$, that is, from the commutativity of R. Sometimes, relations of this form are called "Koszul relations".

We repeat the process for $K^R(x,y,z)=K^R(x,y)\otimes_R K^R(z)$, showing

$$K^R(x,y,z)=\quad 0\to R\xrightarrow{\begin{pmatrix}z\\-y\\x\end{pmatrix}}R^3\xrightarrow{\begin{pmatrix}-y&-z&0\\x&0&-z\\0&x&y\end{pmatrix}}R^3\xrightarrow{(x\ y\ z)}R\to 0.$$

For all n, we have $K^R(x,y,z)_n=\bigoplus_{i+j=n}K^R(x,y)_i\otimes K^R(z)_j$. As $K^R(x,y)_i=0=K^R(z)_j$ for all $i\neq 0,1,2$ and all $j\neq 0,1$, we have $K^R(x,y,z)_n=0$ for $n\neq 0,1,2,3$. In degree 0,

$$K^R(x,y,z)_0=K^R(x,y)_0\otimes_R K^R(z)_0=R(1\otimes 1)\otimes_R R1\cong R$$

with basis vector $1\otimes 1\otimes 1$. The other modules and bases are computed similarly. We summarize in the following table:

i	$K^R(x,y,z)_i$	ordered basis
3	R^1	$e\otimes f\otimes g$
2	R^3	$e\otimes f\otimes 1$, $e\otimes 1\otimes g$, $1\otimes f\otimes g$
1	R^3	$e\otimes 1\otimes 1$, $1\otimes f\otimes 1$, $1\otimes 1\otimes g$
0	R^1	$1\otimes 1\otimes 1$

In particular, $K^R(x,y,z)$ has the following shape:

$$K^R(x,y,z)=\quad 0\to R\xrightarrow{d_3}R^3\xrightarrow{d_2}R^3\xrightarrow{d_1}R\to 0.$$

It follows that d_3 is a 3×1 matrix, d_2 is a 3×3 matrix, and d_1 is a 1×3 matrix, which we identify from the images of the corresponding basis vectors. First, for d_3:

$$\begin{aligned}d_3(e\otimes f\otimes g)&=\partial_1^{K^R(x,y)}(e\otimes f)\otimes g+(-1)^{|e\otimes f|}e\otimes f\otimes\partial_1^{K^R(z)}(g)\\&=x1\otimes f\otimes g-ye\otimes 1\otimes g+ze\otimes f\otimes 1.\end{aligned}$$

This corresponds to the column vector $\begin{pmatrix} z \\ -y \\ x \end{pmatrix}$, and it follows that d_2 is represented by the matrix $\begin{pmatrix} z \\ -y \\ x \end{pmatrix}$. Next, for d_2:

$$\begin{aligned}
d_2(e\otimes f\otimes 1) &= \partial_1^{K^R(x,y)}(e\otimes f)\otimes 1+(-1)^{|e\otimes f|}e\otimes f\otimes \partial_1^{K^R(z)}(1)\\
&= x1\otimes f\otimes 1 - ye\otimes 1\otimes 1\\
d_2(e\otimes 1\otimes g) &= \partial_1^{K^R(x,y)}(e\otimes 1)\otimes g+(-1)^{|e\otimes 1|}e\otimes 1\otimes \partial_1^{K^R(z)}(g)\\
&= x1\otimes 1\otimes g - ze\otimes 1\otimes 1\\
d_2(1\otimes f\otimes g) &= \partial_1^{K^R(x,y)}(1\otimes f)\otimes g+(-1)^{|1\otimes f|}1\otimes f\otimes \partial_1^{K^R(z)}(g)\\
&= y1\otimes 1\otimes g - z1\otimes f\otimes 1.
\end{aligned}$$

It follows that d_2 is represented by the following matrix $d_2 = \begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}$. For d_1, we have three basis vectors to consider, in order:

$$\begin{aligned}
d_1(e\otimes 1\otimes 1) &= \partial_1^{K^R(x,y)}(e\otimes 1)\otimes 1+(-1)^{|e\otimes 1|}e\otimes 1\otimes \partial_1^{K^R(z)}(1)\\
&= x1\otimes 1\otimes 1\\
d_1(1\otimes f\otimes 1) &= \partial_1^{K^R(x,y)}(1\otimes f)\otimes 1+(-1)^{|1\otimes f|}1\otimes f\otimes \partial_1^{K^R(z)}(1)\\
&= y1\otimes 1\otimes 1\\
d_1(1\otimes 1\otimes g) &= \partial_1^{K^R(x,y)}(1\otimes 1)\otimes g+(-1)^{|1\otimes 1|}1\otimes 1\otimes \partial_1^{K^R(z)}(g)\\
&= z1\otimes 1\otimes 1
\end{aligned}$$

It follows that the map d_1 is represented by the row matrix $(x\ y\ z)$, as desired. □

10.14 (Sketch of Solution to Exercise 4.12) This is essentially a manifestation of Pascal's Triangle. We proceed by induction on n, where $\mathbf{x} = x_1,\dots,x_n$. The base case $n = 1$ follows from directly from Definition 4.10. Assume now that the result holds for sequences of length $t \geqslant 1$. We prove the it holds for sequences $x_1,\dots,x_t,x_{t+1}$. By definition, we have

$$K^R(x_1,\dots,x_{t+1}) = K^R(x_1,\dots,x_t)\otimes_R K^R(x_{t+1})$$

Since $K^R(x_{t+1})_i = 0$ for all $i \neq 0,1$, it follows that

$$\begin{aligned}
K^R(x_1,\dots,x_{t+1})_m &= K^R(x_1,\dots,x_t)\otimes_R K^R(x_{t+1})\\
&= (K^R(x_1,\dots,x_t)_m\otimes_R K^R(x_{t+1})_0)\bigoplus(K^R(x_1,\dots,x_t)_{m-1}\otimes_R K^R(x_{t+1})_1)\\
&\cong (R^{\binom{t}{m}}\otimes_R R)\bigoplus(R^{\binom{t}{m-1}}\otimes_R R)\\
&\cong R^{\binom{t}{m}+\binom{t}{m-1}}\\
&= R^{\binom{t+1}{m}}
\end{aligned}$$

which is the desired conclusion. □

10.15 (Sketch of Solution to Exercise 4.13) This follows from the commutativity of tensor product

$$K^R(\mathbf{x}) = K^R(x_1) \otimes_R \cdots \otimes_R K^R(x_n) \cong K^R(x_{\sigma(1)}) \otimes_R \cdots \otimes_R K^R(x_{\sigma(n)}) = K^R(\mathbf{x}')$$

which is the desired conclusion. □

10.16 (Sketch of Solution to Exercise 4.17) Set $(-)^* = \mathrm{Hom}_R(-,R)$. We use the following linear algebra facts freely. First, there is an isomorphism $(R^n)^* \cong R^n$. Second, given an R-module homomorphism $R^m \to R^n$ represented by a matrix A, the dual map $(R^n)^* \to (R^m)^*$ is represented by the transpose A^{T}.

Let $x, y, z \in R$. First, we verify that $\mathrm{Hom}_R(K^R(x),R) \cong \Sigma K^R(x)$, by the above linear algebra facts with Exercise 3.6. The complex $K^R(x)$ has the form

$$K^R(x) = \quad 0 \to R \xrightarrow{x} R \to 0$$

concentrated in degrees 0 and 1. Thus, the shifted complex $\Sigma K^R(x)$ is of the form

$$\Sigma K^R(x) = \quad 0 \to R \xrightarrow{-x} R \to 0$$

concentrated in degrees 0 and -1. By Exercise 3.6, the dual $\mathrm{Hom}_R(K^R(x),R)$ is concentrated in degrees 0 and -1, and has the form

$$\mathrm{Hom}_R(K^R(x),R) = \quad 0 \to R \xrightarrow{x} R \to 0.$$

The following diagram shows that these complexes are isomorphic.

$$\begin{tikzcd}
\Sigma K^R(x) \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "-x"] \arrow[d, "-1"'] & R \arrow[r] \arrow[d, "1"] & 0 \\
\mathrm{Hom}_R(K^R(x),R) & 0 \arrow[r] & R \arrow[r, "x"] & R \arrow[r] & 0
\end{tikzcd}$$

Next, we verify the isomorphism $\mathrm{Hom}_R(K^R(x,y),R) \cong \Sigma^2 K^R(x,y)$. By Exercise 4.11, the complex $K^R(x,y)$ is of the form

$$K^R(x,y) = \quad 0 \to R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \to 0$$

concentrated in degrees $0,1,2$. Thus, the shifted complex $\Sigma K^R(x)$ is of the form

$$\Sigma K^R(x,y) = \quad 0 \to R \xrightarrow{\binom{-y}{x}} R^2 \xrightarrow{(x\ y)} R \to 0$$

concentrated in degrees $0,-1,-2$. By Exercise 3.6, the dual $\mathrm{Hom}_R(K^R(x,y),R)$ has the form and is concentrated in degrees 0 and -1

$$\operatorname{Hom}_R(K^R(x,y),R) = \quad 0 \to R \xrightarrow{\binom{x}{y}} R^2 \xrightarrow{(-y\ x)} R \to 0.$$

The following diagram shows that these complexes are isomorphic.

$$\begin{tikzcd}
\Sigma K^R(x,y) \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "\binom{-y}{x}"] \arrow[d, "1"'] & R^2 \arrow[r, "(x\ y)"] \arrow[d, "\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)"'] & R \arrow[r] \arrow[d, "1"] & 0 \\
\operatorname{Hom}_R(K^R(x,y),R) & 0 \arrow[r] & R \arrow[r, "\binom{x}{y}"'] & R^2 \arrow[r, "(-y\ x)"] & R \arrow[r] & 0
\end{tikzcd}$$

Note that we found this isomorphism, as follows. Use the identity in degree 1. Consider the matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ in degree 1, and solve the linear equations needed to make the right-most square commute. Then check that the identity in degree 2 makes the left-most square commute.

Lastly, we verify the isomorphism $\operatorname{Hom}_R(K^R(x,y,z),R) \cong \Sigma^3 K^R(x,y,z)$. By Exercise 4.11, the complex $K^R(x,y,x)$ is of the form

$$K^R(x,y,z) = \quad 0 \to R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R^3 \xrightarrow{(x\ y\ z)} R \to 0$$

concentrated in degrees $0,1,2,3$. The shifted complex $\Sigma^3 K^R(x,y,z)$ is of the form

$$\Sigma^3 K^R(x,y,z) = \quad 0 \to R \xrightarrow{\begin{pmatrix} -z \\ y \\ -x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} R^3 \xrightarrow{(-x\ -y\ -z)} R \to 0$$

concentrated in degrees $0,-1,-2,-3$. By Exercise 3.6, the dual $\operatorname{Hom}_R(K^R(x),R)$ has the form and is concentrated in degrees $0,-1,-2,-3$

$$\operatorname{Hom}_R(K^R(x),R) = \quad 0 \to R \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & x & 0 \\ -z & 0 & x \\ 0 & -z & y \end{pmatrix}} R^3 \xrightarrow{(z\ -y\ x)} R \to 0.$$

The following diagram

$$\begin{tikzcd}
\Sigma^3 K^R(x,y,z) \arrow[d, "\cong"'] & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} -z \\ y \\ -x \end{pmatrix}"] \arrow[d, "1"'] & R^3 \arrow[r, "\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}"] \arrow[d, "\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}"'] & R^3 \arrow[r, "(-x\ -y\ -z)"] \arrow[d, "\begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix}"'] & R \arrow[r] \arrow[d, "1"] & 0 \\
\operatorname{Hom}_R(K^R(x,y,z),R) & 0 \arrow[r] & R \arrow[r, "\begin{pmatrix} x \\ y \\ z \end{pmatrix}"'] & R^3 \arrow[r, "\begin{pmatrix} -y & x & 0 \\ -z & 0 & x \\ 0 & -z & y \end{pmatrix}"'] & R^3 \arrow[r, "(z\ -y\ x)"'] & R \arrow[r] & 0
\end{tikzcd}$$

shows that these complexes are isomorphic. $\square$

10.17 (Sketch of Solution to Exercise 4.21) Let $\mathbf{x} = x_1, \ldots, x_n \in R$.

By definition of $\widetilde{K}^R(\mathbf{x})$, the first differential is given by $d_1(e_i) = x_i$ for $i = 1, \ldots, n$, represented in matrix form by $R^n \xrightarrow{(x_1 \, \cdots \, x_n)} R$. In particular, we have

$$\widetilde{K}^R(x_1) = \quad 0 \to R \xrightarrow{x_1} R \to 0.$$

For $\widetilde{K}^R(x_1, x_2)$ this says that we only have to compute d_2:

$$d_2(e_1 \wedge e_2) = x_1 e_2 - x_2 e_1$$

so we have

$$\widetilde{K}^R(x_1, x_2) = \quad 0 \to R \xrightarrow{\left(\begin{smallmatrix} -x_2 \\ x_1 \end{smallmatrix}\right)} R^2 \xrightarrow{(x_1 \; x_2)} R \to 0.$$

In comparison with Exercise 4.11, this says that

$$\widetilde{K}^R(x, y) = \quad 0 \to R \xrightarrow{\left(\begin{smallmatrix} -y \\ x \end{smallmatrix}\right)} R^2 \xrightarrow{(x \; y)} R \to 0.$$

For $\widetilde{K}^R(x, y, z)$, we specify an ordering on the basis for $\widetilde{K}^R(x, y, z)_2$

$$e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3$$

and we compute:

$$\begin{aligned} d_2(e_1 \wedge e_2) &= x_1 e_2 - x_2 e_1 \\ d_2(e_1 \wedge e_3) &= x_1 e_3 - x_3 e_1 \\ d_2(e_2 \wedge e_3) &= x_2 e_3 - x_3 e_2 \\ d_3(e_1 \wedge e_2 \wedge e_3) &= x_1 e_2 \wedge e_3 - x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2. \end{aligned}$$

In summary, we have the following:

$$K^R(x_1, x_2, x_3) = \quad 0 \to R \xrightarrow{\left(\begin{smallmatrix} x_3 \\ -x_2 \\ x_1 \end{smallmatrix}\right)} R^3 \xrightarrow{\left(\begin{smallmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{smallmatrix}\right)} R^3 \xrightarrow{(x_1 \; x_2 \; x_3)} R \to 0.$$

Again, this compares directly with Exercise 4.11. $\square$

10.18 (Sketch of Solution to Exercise 4.27) In the following multiplication tables, given an element x from the left column and an element y from the top row, the corresponding element in the table is the product xy.

$\bigwedge R^1$	1	e
1	1	e
e	e	0

$\bigwedge R^2$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

$\bigwedge R^3$	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	0	$e_1\wedge e_2$	$e_1\wedge e_3$
e_2	e_2	$-e_1\wedge e_2$	0	$e_2\wedge e_3$
e_3	e_3	$-e_1\wedge e_3$	$-e_2\wedge e_3$	0
$e_1\wedge e_2$	$e_1\wedge e_2$	0	0	$e_1\wedge e_2\wedge e_3$
$e_1\wedge e_3$	$e_1\wedge e_3$	0	$-e_1\wedge e_2\wedge e_3$	0
$e_2\wedge e_3$	$e_2\wedge e_3$	$e_1\wedge e_2\wedge e_3$	0	0
$e_1\wedge e_2\wedge e_3$	$e_1\wedge e_2\wedge e_3$	0	0	0

$\bigwedge R^3$	$e_1\wedge e_2$	$e_1\wedge e_3$	$e_2\wedge e_3$	$e_1\wedge e_2\wedge e_3$
1	$e_1\wedge e_2$	$e_1\wedge e_3$	$e_2\wedge e_3$	$e_1\wedge e_2\wedge e_3$
e_1	0	0	$e_1\wedge e_2\wedge e_3$	0
e_2	0	$-e_1\wedge e_2\wedge e_3$	0	0
e_3	$e_1\wedge e_2\wedge e_3$	0	0	0
$e_1\wedge e_2$	0	0	0	0
$e_1\wedge e_3$	0	0	0	0
$e_2\wedge e_3$	0	0	0	0
$e_1\wedge e_2\wedge e_3$	0	0	0	0

10.19 (Sketch of Solution to Exercise 4.29) We proceed by induction on t.

Base case: $t=2$. The result is trivial if ι is the identity, so assume for the rest of this paragraph that ι is not the identity permutation. Because of our assumptions on ι, in this case it has the cycle-notation form $\iota=(j_1\ \ j_2)$. If $j_1<j_2$, then by definition we have

$$e_{\iota(j_1)}\wedge e_{\iota(j_2)}=e_{j_2}\wedge e_{j_1}=-e_{j_1}\wedge e_{j_2}=\operatorname{sgn}(\iota)e_{j_1}\wedge e_{j_2}.$$

The same logic applies when $j_i>j_2$.

Induction step: Assume that $t\geqslant 3$ and that the result holds for elements of the form $e_{i_1}\wedge\cdots\wedge e_{i_s}$ such that $2\leqslant s<t$. We proceed by cases.

Case 1: $\iota(j_1)=j_1$. In this case, ι describes a permutation of $\{j_2,\ldots,j_t\}$ with the same signum as ι; thus, our induction hypothesis explains the third equality in the following display

$$\begin{aligned}
e_{\iota(j_1)}\wedge e_{\iota(j_2)}\wedge\cdots\wedge e_{\iota(j_t)} &= e_{\iota(j_1)}\wedge(e_{\iota(j_2)}\wedge\cdots\wedge e_{\iota(j_t)})\\
&= e_{j_1}\wedge(e_{\iota(j_2)}\wedge\cdots\wedge e_{\iota(j_t)})\\
&= e_{j_1}\wedge(\operatorname{sgn}(\iota)e_{j_2}\wedge\cdots\wedge e_{j_t})\\
&= \operatorname{sgn}(\iota)e_{j_1}\wedge(e_{j_2}\wedge\cdots\wedge e_{j_t})\\
&= \operatorname{sgn}(\iota)e_{j_1}\wedge e_{j_2}\wedge\cdots\wedge e_{j_t}
\end{aligned}$$

and the other equalities are by definition and the condition $\iota(j_1)=j_1$.

Case 2: ι has the cycle-notation form $\iota=(j_1\ \ j_2)$ and $j_2=\min\{j_2,j_3,\ldots,j_t\}$. In this case, we are trying to show that

$$e_{j_2}\wedge e_{j_1}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t}=-e_{j_1}\wedge e_{j_2}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t}. \qquad (10.19.1)$$

Let α be the permutation of $\{j_3,\ldots,j_t\}$ such that $\alpha(j_3)<\cdots<\alpha(j_t)$. Our assumption on j_2 implies that $j_2<\alpha(j_3)<\cdots<\alpha(j_t)$.

Case 2a: $j_1<j_2<\alpha(j_3)<\cdots<\alpha(j_t)$. Our induction hypothesis explains the second equality in the following sequence

$$\begin{aligned}
e_{j_2}\wedge e_{j_1}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t} &= e_{j_2}\wedge(e_{j_1}\wedge(e_{j_3}\wedge\cdots\wedge e_{j_t}))\\
&= e_{j_2}\wedge(e_{j_1}\wedge(\operatorname{sgn}(\alpha)e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= \operatorname{sgn}(\alpha)e_{j_2}\wedge(e_{j_1}\wedge(e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= \operatorname{sgn}(\alpha)e_{j_2}\wedge(e_{j_1}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)})\\
&= -\operatorname{sgn}(\alpha)e_{j_1}\wedge e_{j_2}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}
\end{aligned}$$

and the remaining equalities are by definition, where the fourth and fifth ones use our Case 2a assumption. Similar logic explains the next sequence:

$$\begin{aligned}
-e_{j_1}\wedge e_{j_2}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t} &= -e_{j_1}\wedge(e_{j_2}\wedge(e_{j_3}\wedge\cdots\wedge e_{j_t}))\\
&= -e_{j_1}\wedge(e_{j_2}\wedge(\operatorname{sgn}(\alpha)e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= -\operatorname{sgn}(\alpha)e_{j_1}\wedge(e_{j_2}\wedge(e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= -\operatorname{sgn}(\alpha)e_{j_1}\wedge e_{j_2}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}.
\end{aligned}$$

This explains equation (10.19.1) in Case 2a.

Case 2b: $j_2<j_1<\alpha(j_3)<\cdots<\alpha(j_t)$. Our induction hypothesis explains the second equality in the following sequence

$$\begin{aligned}
e_{j_2}\wedge e_{j_1}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t} &= e_{j_2}\wedge(e_{j_1}\wedge(e_{j_3}\wedge\cdots\wedge e_{j_t}))\\
&= e_{j_2}\wedge(e_{j_1}\wedge(\operatorname{sgn}(\alpha)e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= \operatorname{sgn}(\alpha)e_{j_2}\wedge(e_{j_1}\wedge(e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= \operatorname{sgn}(\alpha)e_{j_2}\wedge e_{j_1}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}
\end{aligned}$$

and the remaining equalities are by definition, where the fourth one uses our Case 2b assumption. Similar logic explains the next sequence:

$$\begin{aligned}
-e_{j_1}\wedge e_{j_2}\wedge e_{j_3}\wedge\cdots\wedge e_{j_t} &= -e_{j_1}\wedge(e_{j_2}\wedge(e_{j_3}\wedge\cdots\wedge e_{j_t}))\\
&= -e_{j_1}\wedge(e_{j_2}\wedge(\operatorname{sgn}(\alpha)e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= -\operatorname{sgn}(\alpha)e_{j_1}\wedge(e_{j_2}\wedge(e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}))\\
&= -\operatorname{sgn}(\alpha)e_{j_1}\wedge(e_{j_2}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)})\\
&= \operatorname{sgn}(\alpha)e_{j_2}\wedge e_{j_1}\wedge e_{\alpha(j_3)}\wedge\cdots\wedge e_{\alpha(j_t)}.
\end{aligned}$$

This explains equation (10.19.1) in Case 2b.

Case 2c: $j_2<\alpha(j_3)<\cdots<\alpha(j_p)<j_1<\alpha(j_{p+1})<\cdots<\alpha(j_t)$. Similar logic as in the previous cases explain the following sequences

$$\begin{aligned}
&e_{j_2} \wedge e_{j_1} \wedge e_{j_3} \wedge \cdots \wedge e_{j_t}\\
&\qquad = e_{j_2} \wedge (e_{j_1} \wedge (e_{j_3} \wedge \cdots \wedge e_{j_t}))\\
&\qquad = e_{j_2} \wedge (e_{j_1} \wedge (\operatorname{sgn}(\alpha) e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_t)}))\\
&\qquad = \operatorname{sgn}(\alpha) e_{j_2} \wedge (e_{j_1} \wedge (e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_t)}))\\
&\qquad = \operatorname{sgn}(\alpha) e_{j_2} \wedge ((-1)^{p-2} e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_p)} \wedge e_{j_1} \wedge e_{\alpha(j_{p+1})} \wedge \cdots e_{\alpha(j_t)})\\
&\qquad = (-1)^{p-2} \operatorname{sgn}(\alpha) e_{j_2} \wedge e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_p)} \wedge e_{j_1} \wedge e_{\alpha(j_{p+1})} \wedge \cdots e_{\alpha(j_t)}
\end{aligned}$$

$$\begin{aligned}
&-e_{j_1} \wedge e_{j_2} \wedge e_{j_3} \wedge \cdots \wedge e_{j_t}\\
&\qquad = -e_{j_1} \wedge (e_{j_2} \wedge (e_{j_3} \wedge \cdots \wedge e_{j_t}))\\
&\qquad = -e_{j_1} \wedge (e_{j_2} \wedge (\operatorname{sgn}(\alpha) e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_t)}))\\
&\qquad = -\operatorname{sgn}(\alpha) e_{j_1} \wedge (e_{j_2} \wedge (e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_t)}))\\
&\qquad = -\operatorname{sgn}(\alpha) e_{j_1} \wedge (e_{j_2} \wedge e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_t)})\\
&\qquad = -(-1)^{p-1} \operatorname{sgn}(\alpha) e_{j_2} \wedge e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_p)} \wedge e_{j_1} \wedge e_{\alpha(j_{p+1})} \wedge \cdots e_{\alpha(j_t)}\\
&\qquad = (-1)^{p-2} \operatorname{sgn}(\alpha) e_{j_2} \wedge e_{\alpha(j_3)} \wedge \cdots \wedge e_{\alpha(j_p)} \wedge e_{j_1} \wedge e_{\alpha(j_{p+1})} \wedge \cdots e_{\alpha(j_t)}
\end{aligned}$$

This explains equation (10.19.1) in Case 2c.

Case 2d: $j_2 < \alpha(j_3) < \cdots < \alpha(j_t) < j_1$. This case is handled as in Case 2c.

Case 3: ι has the cycle-notation form $\iota = (j_1\ j_z)$ where $j_z = \min\{j_2, j_3, \ldots, j_t\}$. In this case, we are trying to prove the following:

$$\begin{aligned}
&e_{j_z} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_1} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}\\
&\qquad\qquad = -e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_z} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}.
\end{aligned}$$

This follows from the next sequence, where the first and third equalities are by our induction hypothesis

$$\begin{aligned}
&e_{j_z} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_1} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}\\
&\qquad = (-1)^{z-2} e_{j_z} \wedge e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}\\
&\qquad = -(-1)^{z-2} e_{j_1} \wedge e_{j_z} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}\\
&\qquad = -e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_z} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_t}
\end{aligned}$$

and the second equality is from Case 2.

Case 4: ι has the cycle-notation form $\iota = (j_1\ j_q)$ for some $q \geqslant 2$. Set $j_z = \min\{j_2, j_3, \ldots, j_t\}$.

Case 4a: $j_1 < j_z$. In this case, we have $j_1 = \min\{j_2, \ldots, j_{q-1}, j_1, j_{q+1}, \ldots, j_t\}$, so Case 3 implies that

$$\begin{aligned}
&e_{j_q} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_1} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t}\\
&\qquad\qquad = -e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_q} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t}.
\end{aligned}$$

which is the desired equality in this case.

Case 4b: $j_z < j_1$. In this case, we have

$$\min\{j_2, j_3, \dots, j_t\} = j_z = \min\{j_2, \dots, j_{q-1}, j_1, j_{q+1}, \dots, j_t\}$$

so Case 3 explains the first and third equalities in the next sequence

$$\begin{aligned} &e_{j_q} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_z} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_1} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t} \\ &\qquad = -e_{j_z} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_q} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_1} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t} \\ &\qquad = e_{j_z} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_1} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_q} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t} \\ &\qquad = -e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{z-1}} \wedge e_{j_z} \wedge e_{j_{z+1}} \wedge \cdots \wedge e_{j_{q-1}} \wedge e_{j_q} \wedge e_{j_{q+1}} \wedge \cdots \wedge e_{j_t} \end{aligned}$$

and Case 1 explains the second equality.[13] This is the desired conclusion here.

Case 5: the general case. Cases 1–4 show that the desired result holds for any transposition that fixes $C := \{1, \dots, n\} \setminus \{j_1, \dots, j_t\}$. In general, since ι fixes C, it is a product $\iota = \tau_1 \cdots \tau_m$ of transpositions that fix C. (For instance, this can be seen by decomposing ι, considered as an element of S_t.) Since the result holds for each τ_i, induction on m shows that the desired result holds for ι. The main point is the following: if the result holds for permutations δ and σ that fix C, then

$$\begin{aligned} e_{\delta\sigma(j_1)} \wedge \cdots \wedge e_{\delta\sigma(j_t)} &= \operatorname{sgn}(\delta) e_{\sigma(j_1)} \wedge \cdots \wedge e_{\sigma(j_t)} \\ &= \operatorname{sgn}(\delta) \operatorname{sgn}(\sigma) e_{j_1} \wedge \cdots \wedge e_{j_t} \\ &= \operatorname{sgn}(\delta\sigma) e_{j_1} \wedge \cdots \wedge e_{j_t} \end{aligned}$$

so the result holds for $\delta\sigma$. □

10.20 (Sketch of Solution to Exercise 4.30) Eventually, we will argue by induction on s. To remove technical issues from the induction argument, we deal with some degenerate cases first. If $i_p = i_q$ for some $p < q$, then by definition, we have

$$\begin{aligned} &(e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \cdots \wedge e_{i_q} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) \\ &\qquad = 0 \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) \\ &\qquad = 0 \\ &\qquad = e_{i_1} \wedge \cdots \wedge e_{i_p} \wedge \cdots \wedge e_{i_q} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}. \end{aligned}$$

If $j_p = j_q$ for some $p < q$, then the same logic applies.

For the rest of the proof, we assume that $i_p \neq i_q$ and $j_p \neq j_q$ for all $p < q$. We argue by induction on s.

Base case: $s = 1$. There are two cases to consider. If $i = j_q$ for some q, then Definitions 4.24 and 4.28 imply that

$$e_{i_1} \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) = 0 = e_{i_1} \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.$$

[13] This visual argument assumes that $z < q$. The case $q < z$ is handled similarly, and the case $z = q$ follows from Case 3.

If $i \neq j_q$ for all q, then Definition 4.28 gives the desired equality directly

$$e_{i_1} \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) = e_{i_1} \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.$$

Induction step: assume that $s \geqslant 2$ and that

$$(e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{s-1}}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) = e_{k_1} \wedge e_{k_2} \wedge \cdots \wedge e_{k_{s-1}} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}$$

for all sequences $k_1, \ldots, k_{s-1}$ of distinct elements in $\{1, \ldots, n\}$. Let α be the permutation of $\{i_1, \ldots, i_s\}$ such that $\alpha(i_1) < \cdots < \alpha(i_s)$, and let β be the permutation of $\{j_1, \ldots, j_t\}$ such that $\beta(j_1) < \cdots < \beta(j_t)$. Exercise 4.29 explains the first step in the next sequence, and the second equality is from Definition 4.24:

$$\begin{aligned}
&(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) \\
&\qquad = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)(e_{\alpha(i_1)} \wedge \cdots \wedge e_{\alpha(i_s)}) \wedge (e_{\beta(j_1)} \wedge \cdots \wedge e_{\beta(j_t)}) \\
&\qquad = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)e_{\alpha(i_1)} \wedge [(e_{\alpha(i_2)} \wedge \cdots \wedge e_{\alpha(i_s)}) \wedge (e_{\beta(j_1)} \wedge \cdots \wedge e_{\beta(j_t)})] \\
&\qquad = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)e_{\alpha(i_1)} \wedge (e_{\alpha(i_2)} \wedge \cdots \wedge e_{\alpha(i_s)} \wedge e_{\beta(j_1)} \wedge \cdots \wedge e_{\beta(j_t)}).
\end{aligned}$$

The third equality follows from our induction hypothesis. If $i_p = j_q$ for some p and q, then $\alpha(i_{p'}) = \beta(j_{q'})$ for some p' and q', so the first paragraph of this proof (or the base case, depending on the situation) implies that

$$\begin{aligned}
&(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) \\
&\qquad = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)e_{\alpha(i_1)} \wedge (e_{\alpha(i_2)} \wedge \cdots \wedge e_{\alpha(i_s)} \wedge e_{\beta(j_1)} \wedge \cdots \wedge e_{\beta(j_t)}) \\
&\qquad = 0 \\
&\qquad = e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.
\end{aligned}$$

Thus, we assume that $i_p \neq j_q$ for all p and q. In particular, the permutations α and β combine to give a permutation γ of $\{i_1, \ldots, i_s, j_1, \ldots, j_t\}$ given by the rules $\gamma(i_p) = \alpha(i_p)$ and $\gamma(j_q) = \beta(j_q)$ for all p and q. Furthermore, one has $\operatorname{sgn}(\gamma) = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)$. (To see this, write α as a product $\alpha = \tau_1 \cdots \tau_u$ of transpositions on $\{i_1, \ldots, i_s\}$, write β as a product $\beta = \pi_1 \cdots \pi_u$ of transpositions on $\{j_1, \ldots, j_t\}$, and observe that $\gamma = \tau_1 \cdots \tau_u \pi_1 \cdots \pi_u$ is a product of transpositions on $\{i_1, \ldots, i_s, j_1, \ldots, j_t\}$.) This explains the second equality in the next sequence:

$$\begin{aligned}
&(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) \\
&\qquad = \operatorname{sgn}(\alpha)\operatorname{sgn}(\beta)e_{\alpha(i_1)} \wedge (e_{\alpha(i_2)} \wedge \cdots \wedge e_{\alpha(i_s)} \wedge e_{\beta(j_1)} \wedge \cdots \wedge e_{\beta(j_t)}) \\
&\qquad = \operatorname{sgn}(\gamma)e_{\gamma(i_1)} \wedge (e_{\gamma(i_2)} \wedge \cdots \wedge e_{\gamma(i_s)} \wedge e_{\gamma(j_1)} \wedge \cdots \gamma e_{\beta(j_t)}) \\
&\qquad = \operatorname{sgn}(\gamma)e_{\gamma(i_1)} \wedge e_{\gamma(i_2)} \wedge \cdots \wedge e_{\gamma(i_s)} \wedge e_{\gamma(j_1)} \wedge \cdots \gamma e_{\beta(j_t)} \\
&\qquad = e_{i_1} \wedge \cdots \wedge e_{i_s} \wedge e_{j_1} \wedge \cdots \wedge e_{j_t}.
\end{aligned}$$

The third equality is by our base case. The fourth equality is by Exercise 4.29. $\square$

10.21 (Sketch of Solution to Exercise 4.31) (a) Multiplication in $\bigwedge R^n$ is distributive and unital, by definition. We check associativity first on basis elements, using Exercise 4.30:

$$\begin{aligned}
&[(e_{i_1}\wedge\cdots\wedge e_{i_s})\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t})]\wedge(e_{k_1}\wedge\cdots\wedge e_{k_u})\\
&\qquad= (e_{i_1}\wedge\cdots\wedge e_{i_s}\wedge e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge(e_{k_1}\wedge\cdots\wedge e_{k_u})\\
&\qquad= e_{i_1}\wedge\cdots\wedge e_{i_s}\wedge e_{j_1}\wedge\cdots\wedge e_{j_t}\wedge e_{k_1}\wedge\cdots\wedge e_{k_u}\\
&\qquad= e_{i_1}\wedge\cdots\wedge e_{i_s}\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t}\wedge e_{k_1}\wedge\cdots\wedge e_{k_u})\\
&\qquad= e_{i_1}\wedge\cdots\wedge e_{i_s}\wedge[(e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge(e_{k_1}\wedge\cdots\wedge e_{k_u})].
\end{aligned}$$

The general associativity holds by distributivity: if $\alpha_1,\ldots,\alpha_a,\beta_1,\ldots,\beta_b,\gamma_1,\ldots,\gamma_c$ are basis elements in $\bigwedge R^n$, then we have

$$\begin{aligned}
\left[(\textstyle\sum_i z_i\alpha_i)\wedge\left(\sum_j y_j\beta_j\right)\right]\wedge(\textstyle\sum_k z_k\gamma_k) &= \textstyle\sum_{i,j,k} z_i y_j z_k[(\alpha_i\wedge\beta_j)\wedge\gamma_k]\\
&= \textstyle\sum_{i,j,k} z_i y_j z_k[\alpha_i\wedge(\beta_j\wedge\gamma_k)]\\
&= (\textstyle\sum_i z_i\alpha_i)\wedge\left[\left(\sum_j y_j\beta_j\right)\wedge(\sum_k z_k\gamma_k)\right].
\end{aligned}$$

(b) As in part (a), it suffices to consider basis vectors $\alpha = e_{i_1}\wedge\cdots\wedge e_{i_s}\in\bigwedge^s R^n$ and $\beta = e_{j_1}\wedge\cdots\wedge e_{j_t}\in\bigwedge^t R^n$, and prove that $\alpha\wedge\beta = (-1)^{st}\beta\wedge\alpha$. Note that we are assuming that $i_1<\cdots<i_s$ and $j_1<\cdots<j_t$. If $i_p = j_q$ for some p and q, then Exercise 4.30 implies that

$$\alpha\wedge\beta = 0 = (-1)^{st}\beta\wedge\alpha$$

as desired. Thus, we assume that $i_p\neq j_q$ for all p and q, that is, there are no repetitions in the list $i_1,\ldots,i_s,j_1,\ldots,j_t$.

We proceed by induction on s.

Base case: $s=1$. In this case, the first and third equalities in the next sequence are from Exercise 4.30, using the condition $s=1$:

$$\begin{aligned}
\alpha\wedge\beta &= e_{i_1}\wedge e_{j_1}\wedge\cdots\wedge e_{j_t}\\
&= (-1)^t e_{j_1}\wedge\cdots\wedge e_{j_t}\wedge e_{i_1}\\
&= (-1)^{st}\beta\wedge\alpha.
\end{aligned}$$

The second equality is from Exercise 4.29.

Induction step: assume that $s\geqslant 2$ and that

$$(e_{i_2}\wedge\cdots\wedge e_{i_s})\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t}) = (-1)^{(s-1)t}(e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge(e_{i_2}\wedge\cdots\wedge e_{i_s}).$$

The first, third, and fifth equalities in the next sequence are by associativity:

$$\begin{aligned}
\alpha\wedge\beta &= e_{i_1}\wedge[(e_{i_2}\wedge\cdots\wedge e_{i_s})\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t})]\\
&= e_{i_1}\wedge[(-1)^{(s-1)t}(e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge(e_{i_2}\wedge\cdots\wedge e_{i_s})]\\
&= (-1)^{(s-1)t}[e_{i_1}\wedge(e_{j_1}\wedge\cdots\wedge e_{j_t})]\wedge(e_{i_2}\wedge\cdots\wedge e_{i_s})\\
&= (-1)^{(s-1)t}(-1)^{t}[(e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge e_{i_1}]\wedge(e_{i_2}\wedge\cdots\wedge e_{i_s})\\
&= (-1)^{st}(e_{j_1}\wedge\cdots\wedge e_{j_t})\wedge[e_{i_1}\wedge(e_{i_2}\wedge\cdots\wedge e_{i_s})]\\
&= (-1)^{st}\beta\wedge\alpha.
\end{aligned}$$

The second equality is by our induction hypothesis, the fourth equality is from our base case, and the fifth equality is by Exercise 4.30.

(c) Let $\alpha\in\bigwedge^s R^n$ such that s is odd. If α is a basis vector, then $\alpha\wedge\alpha=0$ by Exercise 4.30. In general, we have $\alpha=\sum_i z_i v_i$ where the v_i are basis vectors in $\bigwedge^s R^n$, and we compute:

$$\begin{aligned}
\alpha\wedge\alpha &= (\textstyle\sum_i z_i v_i)\wedge(\sum_i z_i v_i)\\
&= \textstyle\sum_{i,j} z_i z_j v_i\wedge v_j\\
&= \textstyle\sum_{i=j} z_i z_j v_i\wedge v_j+\sum_{i\neq j} z_i z_j v_i\wedge v_j\\
&= \textstyle\sum_i z_i^2 \underbrace{v_i\wedge v_i}_{=0}+\sum_{i<j} z_i z_j(v_i\wedge v_j+v_j\wedge v_i)\\
&= \textstyle\sum_{i<j} z_i z_j(v_i\wedge v_j+(-1)^{s^2}v_i\wedge v_j)\\
&= \textstyle\sum_{i<j} z_i z_j(v_i\wedge v_j-v_i\wedge v_j)\\
&= 0.
\end{aligned}$$

The vanishing $v_i\wedge v_i=0$ follows from the fact that v_i is a basis vector, so the fifth equality follows from part (b). The sixth equality follows from the fact that s is odd, and the other equalities are routine.

(d) The map $R\to\bigwedge R^n$ is a ring homomorphism since our rule for multiplication $\bigwedge^0 R^n\times\bigwedge^0 R^n\to\bigwedge^0 R^n$ is defined as the usual scalar multiplication $R\times\bigwedge^0 R^n\to\bigwedge^0 R^n$. To see that the image is in the center, let $r\in R=\bigwedge^0 R^n$ and let $z_i\in\bigwedge^i R^n$ for $i=0,\ldots,n$. Using part (b), we have

$$r\wedge(\textstyle\sum_i z_i)=\sum_i(r\wedge z_i)=\sum_i(z_i\wedge r)=(\sum_i z_i)\wedge r$$

so r in the center of $\bigwedge R^n$. $\square$

10.22 (Sketch of Solution to Exercise 4.32) We argue by cases.

Case 1: $j_p=j_q$ for some $p<q$. In this case, we have

$$\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1}\wedge\cdots\wedge e_{j_t})=\partial_i^{\widetilde{K}^R(\mathbf{x})}(0)=0$$

so we need to prove that $\sum_{s=1}^{t}(-1)^{s+1}x_{j_s}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_t}=0$. In this sum, if $s\notin\{p,q\}$, then we have $e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_t}=0$ since the repetition $j_p=j_q$ occurs in this element. This explains the first equality in the next sequence:

$$
\begin{aligned}
&\textstyle\sum_{s=1}^{t}(-1)^{s+1}x_{j_s}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_t}\\
&\quad=(-1)^{p+1}x_{j_p}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_p}}\wedge\cdots\wedge e_{j_t}+(-1)^{q+1}x_{j_p}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_q}}\wedge\cdots\wedge e_{j_t}\\
&\quad=(-1)^{p+1}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}\wedge e_{j_q}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\qquad+(-1)^{q+1}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_p}\wedge\underbrace{e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}}_{q-p-1\text{ factors}}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\quad=(-1)^{p+1}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}\wedge e_{j_q}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\qquad+(-1)^{q+1+q-p-1}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}\wedge e_{j_p}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\quad=(-1)^{p+1}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}\wedge e_{j_q}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\qquad+(-1)^{p}x_{j_p}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge\cdots\wedge e_{j_{q-1}}\wedge e_{j_p}\wedge e_{j_{q+1}}\wedge\cdots\wedge e_{j_t}\\
&\quad=0.
\end{aligned}
$$

The second equality is just a rewriting of the terms, and the third equality is from Exercise 4.29. The fourth and fifth equalities are routine simplification.

Case 2: $j_p \neq j_q$ for all $p \neq q$.

Claim. Given a permutation ι of $j_1,\ldots,j_t$, setting

$$\widehat{\iota}(j_1,\ldots,j_t):=\textstyle\sum_{s=1}^{t}(-1)^{s+1}x_{\iota(j_s)}e_{\iota(j_1)}\wedge\cdots\wedge\widehat{e_{\iota(j_s)}}\wedge\cdots\wedge e_{\iota(j_t)}$$

we have the following where id is the identity permutation.

$$\widehat{\iota}(j_1,\ldots,j_t)=\operatorname{sgn}(\iota)\widehat{\mathrm{id}}(j_1,\ldots,j_t) \tag{10.22.1}$$

To prove this, write ι as a product $\iota=\tau_1\cdots\tau_u$ of adjacent transpositions, that is, $\tau_i=(j_{p_i}\ j_{p_i+1})$ for some $p_i<t$. We argue by induction on u.

Base case: $u=1$. In this case, $\iota=\tau_1$ is a transposition $\iota=(j_p\ j_{p+1})$ for some $p<t$. In this case, we are proving the equality $\widehat{\iota}(j_1,\ldots,j_t)=-\widehat{\mathrm{id}}(j_1,\ldots,j_t)$. We write out the terms of $\widehat{\iota}(j_1,\ldots,j_t)$, starting with the case $s<p$:

$$
\begin{aligned}
&(-1)^{s+1}x_{\iota(j_s)}e_{\iota(j_1)}\wedge\cdots\wedge\widehat{e_{\iota(j_s)}}\wedge\cdots\wedge e_{\iota(j_{p-1})}\wedge e_{\iota(j_p)}\wedge e_{\iota(j_{p+1})}\wedge e_{\iota(j_{p+2})}\wedge\cdots\wedge e_{\iota(j_t)}\\
&\qquad=(-1)^{s+1}x_{j_s}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_{p+1}}\wedge e_{j_p}\wedge e_{j_{p+2}}\wedge\cdots\wedge e_{j_t}\\
&\qquad=-(-1)^{s+1}x_{j_s}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_p}\wedge e_{j_{p+1}}\wedge e_{j_{p+2}}\wedge\cdots\wedge e_{j_t}.
\end{aligned}
$$

This is the sth term of $-\widehat{\mathrm{id}}(j_1,\ldots,j_t)$ in this case. Similar reasoning yields the same conclusion when $s>p+1$. In the case $s=p$, the pth term of $\widehat{\iota}(j_1,\ldots,j_t)$ is

$$
\begin{aligned}
&(-1)^{p+1}x_{\iota(j_p)}e_{\iota(j_1)}\wedge\cdots\wedge e_{\iota(j_{p-1})}\wedge\widehat{e_{\iota(j_p)}}\wedge e_{\iota(j_{p+1})}\wedge e_{\iota(j_{p+2})}\wedge\cdots\wedge e_{\iota(j_t)}\\
&\qquad=(-1)^{p+1}x_{j_{p+1}}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge\widehat{e_{j_{p+1}}}\wedge e_{j_p}\wedge e_{j_{p+2}}\wedge\cdots\wedge e_{j_t}\\
&\qquad=(-1)^{p+1}x_{j_{p+1}}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_p}\wedge\widehat{e_{j_{p+1}}}\wedge e_{j_{p+2}}\wedge\cdots\wedge e_{j_t}\\
&\qquad=-(-1)^{p+2}x_{j_{p+1}}e_{j_1}\wedge\cdots\wedge e_{j_{p-1}}\wedge e_{j_p}\wedge\widehat{e_{j_{p+1}}}\wedge e_{j_{p+2}}\wedge\cdots\wedge e_{j_t}.
\end{aligned}
$$

This is exactly the $p+1$st term of $-\widehat{\mathrm{id}}(j_1,\ldots,j_t)$. Similar reasoning shows that the $p+1$st term of $\widehat{\iota}(j_1,\ldots,j_t)$ is exactly the pth term of $-\widehat{\mathrm{id}}(j_1,\ldots,j_t)$. It follows that equation (10.22.1) holds in the base case.

Induction step: Assume that $u \geqslant 2$ and that the result holds for the permutation $\iota' := \tau_2\cdots\tau_u$. Note that this implies that $\iota = \tau_1\iota'$, so we have $\mathrm{sgn}(\iota) = -\mathrm{sgn}(\iota')$. This explains the first and last equalities in the next sequence:

$$\begin{aligned}
\widehat{\iota}(j_1,\ldots,j_t) &= \widehat{\tau_1\iota'}(j_1,\ldots,j_t)\\
&= \widehat{\tau}_1(\iota'(j_1),\ldots,\iota'(j_t))\\
&= -\widehat{\mathrm{id}}(\iota'(j_1),\ldots,\iota'(j_t))\\
&= -\widehat{\iota'}(j_1,\ldots,j_t)\\
&= -\mathrm{sgn}(\iota')\widehat{\mathrm{id}}(j_1,\ldots,j_t)\\
&= \mathrm{sgn}(\iota)\widehat{\mathrm{id}}(j_1,\ldots,j_t).
\end{aligned}$$

The second and fourth equalities follow easily from the definition of $\widehat{*}$. The third equality is from our base case, and the fifth equality is from our induction hypothesis. This concludes the proof of the claim.

Now we complete the proof of the exercise. Let ι be a permutation of $j_1,\ldots,j_t$ such that $\iota(j_1) < \cdots < \iota(j_t)$. Exercise 4.29 explains the first equality in the next sequence, and the second, third, and fifth equalities are by definition:

$$\begin{aligned}
\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{j_1}\wedge\cdots\wedge e_{j_t}) &= \mathrm{sgn}(\iota)\partial_i^{\widetilde{K}^R(\mathbf{x})}(e_{\iota(j_1)}\wedge\cdots\wedge e_{\iota(j_t)})\\
&= \mathrm{sgn}(\iota)\textstyle\sum_{s=1}^t(-1)^{s+1}x_{\iota(j_s)}e_{\iota(j_1)}\wedge\cdots\wedge\widehat{e_{\iota(j_s)}}\wedge\cdots\wedge e_{\iota(j_t)}\\
&= \mathrm{sgn}(\iota)\widehat{\iota}(j_1,\ldots,j_t)\\
&= \widehat{\mathrm{id}}(j_1,\ldots,j_t)\\
&= \textstyle\sum_{s=1}^t(-1)^{s+1}x_{j_s}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_s}}\wedge\cdots\wedge e_{j_t}
\end{aligned}$$

The fourth equality is by the claim we proved above. $\square$

10.23 (Sketch of Solution to Exercise 4.33) It suffices to verify the formula for basis vectors $\alpha = e_{j_1}\wedge\cdots\wedge e_{j_s}\in\bigwedge^s R^n$ and $\beta = e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}}\in\bigwedge^t R^n$.

$$\begin{aligned}
\partial_{s+t}^{\widetilde{K}^R(\mathbf{x})}(\alpha\wedge\beta) &= \partial_{s+t}^{\widetilde{K}^R(\mathbf{x})}(e_{j_1}\wedge\cdots\wedge e_{j_{s+t}})\\
&= \textstyle\sum_{u=1}^{s+t}(-1)^{u+1}x_{i_u}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_u}}\wedge\cdots\wedge e_{j_{s+t}}\\
&= \textstyle\sum_{u=1}^{s}(-1)^{u+1}x_{i_u}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_u}}\wedge\cdots\wedge e_{j_s}\wedge e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}}\\
&\quad+\textstyle\sum_{u=s+1}^{s+t}(-1)^{u+1}x_{i_u}e_{j_1}\wedge\cdots\wedge e_{j_s}\wedge e_{j_{s+1}}\wedge\cdots\wedge\widehat{e_{j_u}}\wedge\cdots\wedge e_{j_{s+t}}\\
&= \textstyle\sum_{u=1}^{s}(-1)^{u+1}x_{i_u}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_u}}\wedge\cdots\wedge e_{j_s}\wedge e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}}\\
&\quad+\textstyle\sum_{u=1}^{t}(-1)^{u-s+1}x_{i_{s+u}}e_{j_1}\wedge\cdots\wedge e_{j_s}\wedge e_{j_{s+1}}\wedge\cdots\wedge\widehat{e_{j_{s+u}}}\wedge\cdots\wedge e_{j_{s+t}}
\end{aligned}$$

The first and second equalities are by definition, and the third equality is obtained by splitting the sum. The fourth equality is by reindexing. Thus, the first equality in the next sequence is by distributivity.

$$\begin{aligned}
&\partial_{s+t}^{\widetilde{K}^R(\mathbf{x})}(\alpha\wedge\beta)\\
&\quad=\left[\textstyle\sum_{u=1}^{s}(-1)^{u+1}x_{i_u}e_{j_1}\wedge\cdots\wedge\widehat{e_{j_u}}\wedge\cdots\wedge e_{j_s}\right]\wedge(e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}})\\
&\qquad+(-1)^s(e_{j_1}\wedge\cdots\wedge e_{j_s})\wedge\left[\textstyle\sum_{u=1}^{t}(-1)^{u+1}x_{i_{s+u}}e_{j_{s+1}}\wedge\cdots\wedge\widehat{e_{j_{s+u}}}\wedge\cdots\wedge e_{j_{s+t}}\right]\\
&\quad=\partial_s^{\widetilde{K}^R(\mathbf{x})}(e_{j_1}\wedge\cdots\wedge e_{j_s})\wedge(e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}})\\
&\qquad+(-1)^s(e_{j_1}\wedge\cdots\wedge e_{j_s})\wedge\partial_t^{\widetilde{K}^R(\mathbf{x})}(e_{j_{s+1}}\wedge\cdots\wedge e_{j_{s+t}})\\
&\quad=\partial_s^{\widetilde{K}^R(\mathbf{x})}(\alpha)\wedge\beta+(-1)^s\alpha\wedge\partial_t^{\widetilde{K}^R(\mathbf{x})}(\beta).
\end{aligned}$$

The second equality is by Exercise 4.32, and the last equality is by definition. □

10.24 (Sketch of Solution to Exercise 5.4) The fact that multiplication in A is distributive implies that the map $\mu^A\colon A\otimes_R A\to A$ given by $\mu^A(a\otimes b)=ab$ is well-defined and R-linear. To see that it is a chain map, we compute:

$$\begin{aligned}
\partial^A_{|a|+|b|}(\mu^A_{|a|+|b|}(a\otimes b))&=\partial^A_{|a|+|b|}(ab)\\
&=\partial^A_{|a|}(a)b+(-1)^{|a|}a\partial^A_{|b|}(b)\\
&=\mu^A_{|a|+|b|-1}(\partial^A_{|a|}(a)\otimes b+(-1)^{|a|}a\otimes\partial^A_{|b|}(b))\\
&=\mu^A_{|a|+|b|-1}(\partial^{A\otimes_R A}_{|a|+|b|}(a\otimes b)).
\end{aligned}$$

Since the multiplication on A maps $A_0\times A_0\to A_{0+0}=A_0$, we conclude that A_0 is closed under multiplication. Also, the fact that A_0 is an R-module implies that it is closed under addition and subtraction. Thus, the fact that A_0 is a commutative ring can be shown by restricting the axioms of A to A_0. To show that A_0 is an R-algebra, we define a map $R\to A_0$ by $r\mapsto r1_A$. Since A_0 is a ring and an R-module, it is routine to show that this is a ring homomorphism, so A_0 is an R-algebra. □

10.25 (Sketch of Solution to Exercise 5.6) (a) It is straightforward to show that the map $R\to A$ given by $r\mapsto r\cdot 1_A$ respects multiplication. The fact that it is a morphism of DG R-algebras follows from the commutativity of the next diagram

$$\begin{tikzcd}
R= & \cdots \arrow[r] & 0 \arrow[r]\arrow[d] & 0 \arrow[r]\arrow[d] & R \arrow[r]\arrow[d] & 0\arrow[d]\\
A= & \cdots \arrow[r] & A_2 \arrow[r] & A_1 \arrow[r] & A_0 \arrow[r] & 0
\end{tikzcd}$$

which is easily checked.

(b) Argue as in part (a).

(c) With Exercise 3.10 and Lemma 4.18, the essential point is the commutativity of the following diagram

$$\begin{tikzcd}
K = & \cdots \arrow[r] & R^{\binom{n}{2}} \arrow[r] \arrow[d] & R^n \arrow[r, "(x_1\ \dots\ x_n)"] \arrow[d] & R \arrow[r] \arrow[d] & 0 \arrow[d] \\
R/(\mathbf{x}) = & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & R/(\mathbf{x}) \arrow[r] & 0
\end{tikzcd}$$

which is easily checked. □

10.26 (Sketch of Solution to Exercise 5.7) (a) The condition $A_{-1} = 0$ implies that $\mathrm{H}_0(A) \cong A_0/\mathrm{Im}(\partial_1^A)$, which is a homomorphic image of A_0. To show that $\mathrm{H}_0(A)$ is an A_0-algebra, it suffices to show that $\mathrm{Im}(\partial_1^A)$ is an ideal of A_0. To this end, the fact that ∂_1^A is R-linear implies that $\mathrm{Im}(\partial_1^A)$ is non-empty and closed under subtraction. To show that it is closed under multiplication by elements of A_0, let $a_0 \in A_0$ and $\partial_1^A(a_1) \in \mathrm{Im}(\partial_1^A)$, and use the Leibniz Rule

$$\partial_1^A(a_0a_1) = \partial_0^A(a_0)a_1 + (-1)^0 a_0\partial_1^A(a_1) = a_0\partial_1^A(a_1)$$

to see that $a_0\partial_1^A(a_1) \in \mathrm{Im}(\partial_1^A)$.

(b) To see that A_i is an A_0-module, first observe that multiplication in A maps $A_0 \times A_i$ to A_i. Thus, A_i is closed under scalar multiplication by A_0. Since A_i is an R-module, it is non-empty and closed under addition and subtraction. The remaining A_0-module axioms follow from the DG algebra axioms on A.

To show that $\mathrm{H}_i(A)$ is an $\mathrm{H}_0(A)$-module, the essential point is to show that the scalar multiplication $\overline{a_0}\ \overline{a_i} := \overline{a_0a_i}$ is well-defined. (Then the axioms follow directly from the fact that A_i is an A_0-module.) The well-definedness boils down to showing that the products $\mathrm{Im}(\partial_1^A)\,\mathrm{Ker}(\partial_i^A)$ and $\mathrm{Ker}(\partial_0^A)\,\mathrm{Im}(\partial_{i+1}^A)$ are contained in $\mathrm{Im}(\partial_{i+1}^A)$. For the first of these, let $\partial_1^A(a_1) \in \mathrm{Im}(\partial_1^A)$ and $z_i \in \mathrm{Ker}(\partial_i^A)$:

$$\partial_{i+1}^A(a_1z_i) = \partial_1^A(a_1)z_i + (-1)^1 a_1 \underbrace{\partial_i^A(z_i)}_{=0} = \partial_1^A(a_1)z_i.$$

It follows that $\partial_1^A(a_1)z_i = \partial_{i+1}^A(a_1z_i) \in \mathrm{Im}(\partial_{i+1}^A)$, as desired. The containment $\mathrm{Ker}(\partial_0^A)\,\mathrm{Im}(\partial_{i+1}^A) \subseteq \mathrm{Im}(\partial_{i+1}^A)$ is verified similarly. □

10.27 (Sketch of Solution to Exercise 5.10) Assume that A is a DG R-algebra such that each A_i is finitely generated. Since R is noetherian, each A_i is noetherian over R. Hence, each submodule $\mathrm{Ker}(\partial_i^A) \subseteq A_i$ is finitely generated over R, so each quotient $\mathrm{H}_i(A) = \mathrm{Ker}(\partial_i^A)/\mathrm{Im}(\partial_{i+1}^A)$ is finitely generated over R. In particular, $\mathrm{H}_0(A)$ is finitely generated over R, so it is noetherian by the Hilbert Basis Theorem. Since $\mathrm{H}_i(A)$ is finitely generated over R and it is a module over the R-algebra $\mathrm{H}_0(A)$, it is straightforward to show that $\mathrm{H}_i(A)$ is finitely generated over $\mathrm{H}_0(A)$. □

10.28 (Sketch of Solution to Exercise 5.12) (a) By definition, every DG R-module is, in particular, an R-complex. Conversely, let X be an R-complex. We verify the

DG R-module axioms. The associative, distributive, and unital axioms are automatic since X is an R-complex.

For the graded axiom, let $r \in R_i$ and $x \in X_j$. If $i \neq 0$, then $r \in R_i = 0$ implies that $rx = 0 \in X_{i+j}$. If $i = 0$, then $r \in R_i = R$ implies that $rx \in X_j = X_{0+j}$.

For the Leibniz Rule, let $r \in R_i$ and $x \in X_j$. If $i \neq 0$, then $r \in R_i = 0$ implies

$$\partial^X_{i+j}(rx) = \partial^X_{i+j}(0) = 0 = \partial^X_{i+j}(0)x + (-1)^i 0 \partial^X_{i+j}(x) = \partial^X_i(r)x + (-1)^i r \partial^X_j(x)$$

as desired. If $i = 0$, then $r \in R_i = R$, so the R-linearity of $\partial^X_j = \partial^X_{0+j}$ implies that

$$\partial^X_{i+j}(rx) = r\partial^X_{0+j}(x) = 0x + (-1)^0 r\partial^X_j(x) = \partial^X_i(r)x + (-1)^i r\partial^X_j(x)$$

as desired.

(b) Directly compare the axioms in Definitions 5.1 and 5.11.

(c) Let $f\colon A \to B$ a morphism of DG R-algebras, and let Y be a DG B-module. Define the DG A-module structure on Y by the formula $a_i y_j := f_i(a_i) y_j$.

We verify associativity:

$$(a_i a_j) y_k = f_{i+j}(a_i a_j) y_k = [f_i(a_i) f_j(a_j)] y_k = f_i(a_i)[f_j(a_j) y_k] = a_i(a_j y_k).$$

Distributivity, gradedness, and the Leibniz Rule follow similarly, as does unitality (using the condition $f_0(1_A) = 1_B$). $\square$

10.29 (Sketch of Solution to Exercise 5.13) Fix an R-complex X and a DG R-algebra A. First, we show that the scalar multiplication $a(b \otimes x) := (ab) \otimes x$ is well-defined. Let $a_i \in A_i$ be a fixed element. Since the map $A_j \to A_{i+j}$ given by $a_j \mapsto a_i a_j$ is well-defined and R-linear, so is the induced map $A_j \otimes_R X_k \to A_{i+j} \otimes_R X_k$ which is given on generators by the formula $a_j \otimes x_k \mapsto (a_i a_j) \otimes x_k$. Assembling these maps together for all j, k provides a well-defined R-linear map

$$(A \otimes_R X)_n = \bigoplus_{j+k=n} A_j \otimes_R X_k \to \bigoplus_{j+k=n} A_{i+j} \otimes_R X_k = (A \otimes_R X)_{i+n}$$

given on generators by the formula $a_j \otimes x_k \mapsto (a_i a_j) \otimes x_k$. In other words, the given multiplication is well-defined and satisfies the graded axiom. Verification of associativity and distributivity is routine. We verify the Leibniz Rule on generators:

$$\begin{aligned}
&\partial^{A\otimes_R X}_{i+j+k}(a_i(a_j \otimes x_k)) \\
&\qquad = \partial^{A\otimes_R X}_{i+j+k}((a_i a_j) \otimes x_k) \\
&\qquad = \partial^A_{i+j}(a_i a_j) \otimes x_k + (-1)^{i+j}(a_i a_j) \otimes \partial^X_k(x_k) \\
&\qquad = [\partial^A_i(a_i) a_j] \otimes x_k + (-1)^i [a_i \partial^A_j(a_j)] \otimes x_k + (-1)^{i+j}(a_i a_j) \otimes \partial^X_k(x_k) \\
&\qquad = \partial^A_i(a_i)(a_j \otimes x_k) + (-1)^i a_i [\partial^A_j(a_j) \otimes x_k + (-1)^j a_j \otimes \partial^X_k(x_k)] \\
&\qquad = \partial^A_i(a_i)(a_j \otimes x_k) + (-1)^i a_i \partial^{A\otimes_R X}_{j+k}(a_j \otimes x_k).
\end{aligned}$$

$\square$

10.30 (Sketch of Solution to Exercise 5.14) Since scalar multiplication of A on M is well-defined, so is the scalar multiplication of A on $\Sigma^i M$. To prove that $\Sigma^i M$ is a DG A-module, we check the axioms, in order, beginning with associativity:

$$\begin{aligned} a*(b*m) &= (-1)^{i|a|}a((-1)^{i|b|}bm) = (-1)^{i(|a|+|b|)}a(bm) \\ &= (-1)^{i|ab|}(ab)m = (ab)*m. \end{aligned}$$

Distributivity is verified similarly. For the unital axiom, recall that $|1_A| = 0$, so we have $1_A * m = (-1)^{i\cdot 0} 1_A m = m$.

The graded axiom requires a bit of bookkeeping. Let $a \in A_p$ and $m \in (\Sigma^i M)_q = M_{q-i}$. Then we have $a * m = (-1)^{ip} am \in M_{p+q-i} = (\Sigma^i M)_{p+q}$, as desired.

For the Leibniz Rule, we let $a \in A_p$ and $m \in (\Sigma^i M)_q = M_{q-i}$, and we compute:

$$\begin{aligned} \partial^{\Sigma^i M}_{p+q}(a*m) &= (-1)^i \partial^M_{p+q-i}((-1)^{ip}am) \\ &= (-1)^{ip+i}[\partial^A_p(a)m + (-1)^p a\partial^M_{q-i}(m)] \\ &= (-1)^{ip-i}[\partial^A_p(a)m + (-1)^{p+i} a\partial^{\Sigma^i M}_q(m)] \\ &= (-1)^{i(p-1)}\partial^A_p(a)m + (-1)^{p+ip} a\partial^{\Sigma^i M}_q(m) \\ &= \partial^A_p(a)*m + (-1)^p a*\partial^{\Sigma^i M}_q(m). \end{aligned}$$

□

10.31 (Sketch of Solution to Exercise 5.15) As for Exercise 5.4; cf. 10.24. □

10.32 (Sketch of Solution to Exercise 5.22) Fix a DG R-algebra A and a chain map of R-complexes $g\colon X \to Y$.

(a) To prove that the chain map $A \otimes_R g\colon A \otimes_R X \to A \otimes_R Y$ is a morphism of DG A-modules, we check the desired formula on generators:

$$\begin{aligned} (A \otimes_R g)_{i+j+k}(a_i(a_j \otimes x_k)) &= (A \otimes_R g)_{i+j+k}((a_i a_j) \otimes x_k)) \\ &= (a_i a_j) \otimes g_k(x_k) \\ &= a_i(a_j \otimes g_k(x_k)) \\ &= a_i(A \otimes_R g)_{j+k}(a_j \otimes x_k). \end{aligned}$$

(b) Let k be a field, and consider the power series ring $R = k[\![X]\!]$. The residue field k is an R-algebra, hence it is a DG R-algebra. The Koszul complex $K = K^R(X)$ is a free resolution of k over R, so the natural map $K \to k$ is a quasiisomorphism. On the other hand, the induced map $k \otimes_R K \to k \otimes_R k$ is not a quasiisomorphism since $\mathrm{H}_1(k \otimes_R K) \cong k \neq 0 = \mathrm{H}_i(k \otimes_R k)$.

(c) Fix a morphism $f\colon A \to B$ of DG R-algebras. This explains the first equality in the next display

$$f(ab) = f(a)f(b) = af(b).$$

The second equality is from the definition of the DG A-module structure on B. □

10.33 (Sketch of Solution to Exercise 5.27) We use the following items.

Definition 10.34 Let X be an R-complex. An R-*subcomplex* of X is an R-complex Y such that each Y_i is a submodule of X_i and for all $y \in Y_i$ we have $\partial_i^Y(y) = \partial_i^X(y)$. A *DG submodule* of the DG A-module M is an R-subcomplex N such that the scalar multiplication of A on N uses the same rule as the scalar multiplication on M.

Lemma 10.35 *Let X be an R-complex, and let Y be an R-subcomplex of X.*

(a) *The quotient complex X/Y with differential induced by ∂^X is an R-complex.*
(b) *The natural sequence $0 \to Y \xrightarrow{\varepsilon} X \xrightarrow{\pi} X/Y \to 0$ of chain maps is exact.*
(c) *The natural map $X \to X/Y$ is a quasiisomorphism if and only if Y is exact.*

Proof. (a) To show that the differential $\partial^{X/Y}$ given as $\partial_i^{X/Y}(\overline{x}) := \overline{\partial_i^X(x)}$ is well-defined, it suffices to show that $\partial_i^X(Y_i) \subseteq Y_{i-1}$. But this condition is automatic by definition: for all $y \in Y_i$ we have $\partial_i^X(y) = \partial_i^Y(y) \in Y_{i-1}$. The remaining R-complex axioms are straightforward consequences of the axioms for X.

(b) It suffices to show that the following diagram commutes

$$\begin{tikzcd}
0 \arrow[r] & Y_i \arrow[r, "\varepsilon_i"] \arrow[d, "\partial_i^Y"'] & X_i \arrow[r, "\pi_i"] \arrow[d, "\partial_i^X"'] & X_i/Y_i \arrow[r] \arrow[d, "\partial_i^{X/Y}"'] & 0 \\
0 \arrow[r] & Y_{i-1} \arrow[r, "\varepsilon_{i-1}"] & X_{i-1} \arrow[r, "\pi_{i-1}"] & X_{i-1}/Y_{i-1} \arrow[r] & 0
\end{tikzcd}$$

where ε_i is the inclusion and π_i is the natural surjection. The commutativity of this diagram is routine, using the assumption on ∂^Y and the definition of $\partial^{X/Y}$.

(c) Use the long exact sequence coming from part (b). $\square$

Lemma 10.36 *Let N be a DG submodule of the DG A-module M.*

(a) *The quotient complex M/N with scalar multiplication induced by the scalar multiplication on M is a DG A-module.*
(b) *The natural maps $N \xrightarrow{\varepsilon} M \xrightarrow{\pi} M/N$ are morphisms of DG A-modules.*

Proof. (a) Lemma 10.35(a) implies that M/N is an R-complex. Next, we show that the scalar multiplication $a\overline{m} := \overline{am}$ is well-defined. For this, it suffices to show that $an \in N$ for all $a \in A$ and all $n \in N$. But this condition is automatic by definition, since N is closed under scalar multiplication. Now that the scalar multiplication on M/N is well-defined, the DG A-module axioms on M/N follow readily from the axioms on M.

(b) As N is a DG submodule of M, the inclusion ε respects scalar multiplication; and π respects scalar multiplication as follows: $\pi(am) = \overline{am} = a\overline{m} = a\pi(m)$. $\square$

Now, we continue with the proof of the exercise. Consider the complex

$$N = \cdots \xrightarrow{\partial^M_{n+2}} M_{n+1} \xrightarrow{\partial^M_{n+1}} \operatorname{Im}(\partial^M_{n+1}) \to 0.$$

(a) Using Lemmas 10.35 and 10.36, it suffices to show that N is a DG submodule of M. By inspection, the differential on N maps $N_i \to N_{i-1}$ for all i. Since the differential and scalar multiplication on N are induced from those on M, it suffices to show that N is closed under scalar multiplication. (The other axioms are inherited from M.) For this, let $a \in A_p$ and $x \in N_q$. If $p < 0$, then $a = 0$ and we have $ax = 0 \in N_{p+q}$. Similar reasoning applies if $q < n$. Assume now that $p \geqslant 0$ and $q \geqslant n$. If $p \geqslant 1$ or $q > n$, then $ax \in M_{p+q} = N_{p+q}$, by definition. So, we are reduced to the case where $p = 0$ and $q = n$. In this case, there is an element $m \in M_{n+1}$ such that $x = \partial^M_{n+1}(m)$. Thus, the Leibniz Rule on M implies that

$$ax = 0 + ax = \partial^A_0(a)m + a\partial^M_{n+1}(m) = \partial^M_{n+1}(am) \in \operatorname{Im}(\partial^M_{n+1}) = N_n$$

as desired.

(b) Using Lemmas 10.35 and 10.36, it suffices to show that N is exact if and only if $n \geqslant \sup M$. For this, note that

$$\mathrm{H}_i(N) \cong \begin{cases} \mathrm{H}_i(M) & \text{for } i > n \\ 0 & \text{for } i \leqslant n. \end{cases}$$

It follows that the complex N is exact if and only if $\mathrm{H}_i(M) = 0$ for all $i > n$, that is, if and only if $n \geqslant \sup(M)$. $\square$

10.37 (Sketch of Solution to Exercise 6.4) Note that $I = (x^2, xy, y^2)$ has grade two as an ideal of R since x^2, y^2 is an R-regular sequence. Also, $I = I_2(A)$, where

$$A = \begin{bmatrix} x & 0 \\ y & x \\ 0 & y \end{bmatrix}$$

Thus, I is perfect by Theorem 6.3. $\square$

10.38 (Sketch of Solution to Exercise 6.6) Let $A = (a_{i,j})$, which is $(n+1) \times n$. Let F denote the R-complex in question, and denote the differential of F by ∂. First, we check the easy products. The fact that $F_3 = 0$ explains the next sequence.

$$\partial_2(f_j)f_k + (-1)^2 f_j\partial_2(f_k) = 0 = \partial_4(f_jf_k)$$

Next, we argue by definition and cancellation.

$$\partial_1(e_i)e_i + (-1)^1 e_i\partial_1(e_i) = (-1)^{i-1}a\det(A_i)e_i - e_i[(-1)^{i-1}a\det(A_i)] = 0 = \partial_2(e_ie_i)$$

The remaining products require some work.

Let $1 \leqslant i < j \leqslant n+1$, and consider the $n \times n$ matrices A_j and A_i. Expanding $\det(A_j)$ along the ith row, we have

$$\det(A_j) = \sum_{k=1}^{n} (-1)^{i+k} a_{i,k} \det(A_{i,j}^k). \tag{10.38.1}$$

This uses the equality $(A_j)_i^k = A_{i,j}^k$ which is a consequence of the assumption $i < j$. On the other hand, we have $(A_i)_{j-1}^k = A_{i,j}^k$ since the $(j-1)$st row of A_i is equal to the jth row of A; so when we expand $\det(A_i)$ along its $(j-1)$st row, we have

$$\det(A_i) = \sum_{k=1}^{n} (-1)^{j-1+k} a_{j,k} \det((A_i)_{j-1}^k) = \sum_{k=1}^{n} (-1)^{j-1+k} a_{j,k} \det(A_{i,j}^k). \tag{10.38.2}$$

Next, for $\ell \neq i, j$ we let $A(\ell)$ denote the matrix obtained by replacing the ith row of A with the ℓth row. It follows that we have $(A(\ell)_j)_i^k = A(\ell)_{i,j}^k = A_{i,j}^k$. Notice that the matrix $A(\ell)_j$ has two equal rows, so we have $\det(A(\ell)_j) = 0$. Expanding $\det(A(\ell)_j)$ along the ith row, we obtain the next equalities:

$$0 = \sum_{k=1}^{n} (-1)^{i+k} a_{\ell,k} \det((A(\ell)_j)_i^k) = \sum_{k=1}^{n} (-1)^{i+k} a_{\ell,k} \det(A_{i,j}^k). \tag{10.38.3}$$

Now we verify the Leibniz rule for the product $e_i e_j$, still assuming $1 \leqslant i < j \leqslant n+1$.

$$\begin{aligned}
\partial_2(e_i e_j) &= a \sum_{k=1}^{n} (-1)^{i+j+k} \det(A_{i,j}^k) \partial_2(f_k) \\
&= a \sum_{k=1}^{n} (-1)^{i+j+k} \det(A_{i,j}^k) \sum_{\ell=1}^{n+1} a_{\ell,k} e_\ell \\
&= a \sum_{\ell=1}^{n+1} \left[\sum_{k=1}^{n} (-1)^{i+j+k} a_{\ell,k} \det(A_{i,j}^k) \right] e_\ell \\
&= a \left[\sum_{k=1}^{n} (-1)^{i+j+k} a_{i,k} \det(A_{i,j}^k) \right] e_i + a \left[\sum_{k=1}^{n} (-1)^{i+j+k} a_{j,k} \det(A_{i,j}^k) \right] e_j \\
&\quad + a \sum_{\ell \neq i,j} \left[\sum_{k=1}^{n} (-1)^{i+j+k} a_{\ell,k} \det(A_{i,j}^k) \right] e_\ell \\
&= (-1)^j a \left[\sum_{k=1}^{n} (-1)^{i+k} a_{i,k} \det(A_{i,j}^k) \right] e_i \\
&\quad + (-1)^{i+1} a \left[\sum_{k=1}^{n} (-1)^{j-1+k} a_{j,k} \det(A_{i,j}^k) \right] e_j \\
&\quad + a \sum_{\ell \neq i,j} \left[\sum_{k=1}^{n} (-1)^{i+j+k} a_{\ell,k} \det(A_{i,j}^k) \right] e_\ell
\end{aligned}$$

The above equalities are by definition and simplification. In the next sequence, the second equality follows from (10.38.1)–(10.38.3), and the others are by definition and simplification.

$$\begin{aligned}
\partial_2(e_ie_j) &= a\sum_{k=1}^{n}(-1)^{i+j+k}\det(A_{i,j}^k)\partial_2(f_k)\\
&= (-1)^j a\det(A_j)e_i + (-1)^{i+1}a\det(A_i)e_j + (-1)^j a\sum_{\ell\neq i,j}0e_\ell\\
&= -\partial_1(e_j)e_i + \partial_1(e_i)e_j + 0\\
&= \partial_1(e_i)e_j + (-1)^{|e_i|}e_i\partial_1(e_j)
\end{aligned}$$

Next, we show how the Leibniz rule for e_je_i follows from that of e_ie_j:

$$\begin{aligned}
\partial_2(e_je_i) = \partial_2(-e_ie_j) = -\partial_2(e_ie_j) &= -[\partial_1(e_i)e_j - e_i\partial_1(e_j)]\\
&= e_i\partial_1(e_j) - \partial_1(e_i)e_j = \partial_1(e_j)e_i - e_j\partial_1(e_i).
\end{aligned}$$

Next, we verify the Leibniz rule for products of the form e_if_j for any i and j. To begin, note that we have

$$(A_i)_k^j = \begin{cases} A_{k,i}^j & \text{if } k < i\\ A_{i,k+1}^j & \text{if } k \geqslant i\end{cases}$$

and the k,j-entry of A_i is

$$a'_{k,j} := \begin{cases} a_{k,j} & \text{if } k < i\\ a_{k+1,j} & \text{if } k \geqslant i.\end{cases}$$

Using this, we expand $\det(A_i)$ along the jth column:

$$\begin{aligned}
\det(A_i) &= \sum_{k=1}^{n}(-1)^{j+k}\det((A_i)_k^j)a'_{k,j}\\
&= \sum_{k=1}^{i-1}(-1)^{j+k}\det(A_{k,i}^j)a_{k,j} + \sum_{k=i}^{n}(-1)^{j+k}\det(A_{i,k+1}^j)a_{k+1,j}\\
&= \sum_{k=1}^{i-1}(-1)^{j+k}\det(A_{k,i}^j)a_{k,j} + \sum_{k=i+1}^{n+1}(-1)^{j+k-1}\det(A_{i,k}^j)a_{k,j}\\
&= \sum_{k=1}^{i-1}(-1)^{j+k}\det(A_{k,i}^j)a_{k,j} - \sum_{k=i+1}^{n+1}(-1)^{j+k}\det(A_{i,k}^j)a_{k,j}.
\end{aligned}\tag{10.38.4}$$

Similarly, for any $\ell \neq j$, let $A[\ell]$ denote the matrix obtained by replacing the ℓth column of A with its jth column. Thus, we have $\det(A[\ell]_i) = 0$. Expanding $\det(A[\ell]_i)$

along the ℓth column as above implies that

$$\sum_{k=1}^{i-1}(-1)^{\ell+k}\det(A^{\ell}_{k,i})a_{k,j}-\sum_{k=i+1}^{n+1}(-1)^{\ell+k}\det(A^{\ell}_{i,k})a_{k,j}=0. \tag{10.38.5}$$

By definition, we have $e_i f_j = 0$. Thus, the Leibniz rule in this case follows from the next computation where the final equality is a consequence of (10.38.4)–(10.38.5):

$$\begin{aligned}
\partial_1(e_i)f_j - e_i\partial_2(f_j) &= (-1)^{i+1}a\det(A_i)f_j - e_i\sum_{k=1}^{n+1}a_{k,j}e_k\\
&= (-1)^{i+1}a\det(A_i)f_j - \sum_{k=1}^{n+1}a_{k,j}e_ie_k\\
&= (-1)^{i+1}a\det(A_i)f_j + \sum_{k=1}^{i-1}a_{k,j}e_ke_i - \sum_{k=i+1}^{n+1}a_{k,j}e_ie_k\\
&= (-1)^{i+1}a\det(A_i)f_j + \sum_{k=1}^{i-1}a_{k,j}a\sum_{\ell=1}^{n}(-1)^{k+i+\ell}\det(A^{\ell}_{k,i})f_\ell\\
&\qquad - \sum_{k=i+1}^{n+1}a_{k,j}a\sum_{\ell=1}^{n}(-1)^{i+k+\ell}\det(A^{\ell}_{i,k})f_\ell\\
&= (-1)^{i+1}a\det(A_i)f_j + a(-1)^i\sum_{\ell=1}^{n}\left[\sum_{k=1}^{i-1}a_{k,j}(-1)^{k+\ell}\det(A^{\ell}_{k,i})\right.\\
&\qquad \left. - \sum_{k=i+1}^{n+1}a_{k,j}(-1)^{k+\ell}\det(A^{\ell}_{i,k})\right]f_\ell\\
&= 0.
\end{aligned}$$

The final case now follows from the previous one:

$$\partial_2(f_j)e_i + f_j\partial_1(e_i) = -e_i\partial_2(f_j)+\partial_1(e_i)f_j = \partial_1(e_i)f_j - e_i\partial_2(f_j) = 0.$$

This completes the proof. $\square$

10.39 (Sketch of Solution to Exercise 6.7) The deleted minimal R-free resolution of $R/(x^2,xy,y^2)$ is the following:

$$0 \to Rf_1\oplus Rf_2 \xrightarrow{\left[\begin{smallmatrix} x & 0\\ y & x\\ 0 & y\end{smallmatrix}\right]} Re_1\oplus Re_2\oplus Re_3 \xrightarrow{[-y^2\ xy\ -x^2]} R1 \to 0.$$

According to Theorem 6.5, the above complex has a DG R-algebra structure with

$$\begin{aligned}
e_1e_2 &= -e_2e_1 = yf_1 \\
e_1e_3 &= -e_3e_1 = -xf_1 + yf_2 \\
e_2e_3 &= -e_3e_2 = -xf_2
\end{aligned}$$

and $e_i^2 = 0$ for all $1 \leqslant i \leqslant 3$. $\square$

10.40 (Sketch of Solution to Exercise 6.15) First, note that since A is a 3×3 matrix, $\mathrm{Pf}(A_{ijk}^{ijk}) = 1$ for all choices of i,j,k. Following the conditions specified by Theorem 6.14, one has the relations $e_i^2 = 0$ for all $1 \leqslant i \leqslant 3$ and

$$\begin{aligned}
e_1e_2 &= -e_2e_1 = f_1 - f_2 + f_3 \\
e_1e_3 &= -e_3e_1 = -f_1 - f_2 - f_3 \\
e_2e_3 &= -e_3e_2 = f_1 - f_2 + f_3
\end{aligned}$$

and $e_if_j = f_je_i = \delta_{ij}g$ for all $1 \leqslant i,j \leqslant 3$. $\square$

10.41 (Sketch of Solution to Exercise 7.3)

(a) To start, we let $f = \{f_j\} \in \mathrm{Hom}_A(M,N)_q$ and prove that $\partial_q^{\mathrm{Hom}_R(M,N)}(f)$ is in $\mathrm{Hom}_A(M,N)_{q-1}$, that is, that $\partial_q^{\mathrm{Hom}_R(M,N)}(f)$ is A-linear. For this, let $a \in A_p$ and $m \in M_t$, and compute:

$$\begin{aligned}
\partial_q^{\mathrm{Hom}_R(M,N)}(f)_{p+t}(am) &= \partial_{p+t+q}^N(f_{p+t}(am)) - (-1)^q f_{p+t-1}(\partial_{p+t}^M(am)) \\
&= (-1)^{pq}\partial_{p+t+q}^N(af_t(m)) \\
&\quad - (-1)^q f_{p+t-1}\left(\partial_p^A(a)m + (-1)^p a\partial_t^M(m)\right) \\
&= (-1)^{pq}\left(\partial_p^A(a)f_t(m) + (-1)^p a\partial_{t+q}^N(f_t(m))\right) \\
&\quad - (-1)^q f_{p+t-1}\left(\partial_p^A(a)m\right) - (-1)^{q+p} f_{p+t-1}\left(a\partial_t^M(m)\right) \\
&= (-1)^{pq}\partial_p^A(a)f_t(m) + (-1)^{pq+p} a\partial_{t+q}^N(f_t(m)) \\
&\quad - (-1)^{q+(p-1)q}\partial_p^A(a)f_t(m) - (-1)^{q+p+pq} af_{t-1}\left(\partial_t^M(m)\right) \\
&= (-1)^{pq+p} a\partial_{t+q}^N(f_t(m)) - (-1)^{q+p+pq} af_{t-1}\left(\partial_t^M(m)\right) \\
&= (-1)^{p(q-1)} a\left(\partial_{t+q}^N(f_t(m)) - (-1)^q f_{t-1}(\partial_t^M(m))\right) \\
&= (-1)^{p(q-1)} a\partial_q^{\mathrm{Hom}_R(M,N)}(f)_{p+t}(m)
\end{aligned}$$

The first and last equalities are from the definition of $\partial_q^{\mathrm{Hom}_R(M,N)}(f)$. The second equality is from the A-linearity of f and the Leibniz Rule on M. The third equality is from the Leibniz Rule on N and the A-linearity of f. The fourth equality is from the A-linearity of f. The fifth equality is by cancellation since $q + (p-1)q = pq$. And the sixth equality is distributivity.

This shows that $\partial^{\mathrm{Hom}_A(M,N)}$ is well-defined. Since $\mathrm{Hom}_R(M,N)$ is an R-complex, it follows readily that $\mathrm{Hom}_A(M,N)$ is also an R-complex.

With the same a and f as above, we next show that the sequence $af := \{(af)_j\}$ defined by the formula $(af)_t(m) := a(f_t(m))$ is in $\mathrm{Hom}_A(M,N)_{p+q}$. First, this rule maps $M_t \to N_{t+p+q}$ since $m \in M_t$ implies that $f_t(m) \in N_{t+q}$, which implies that $a(f_t(m)) \in N_{t+q+p}$. Next, we show that $af \in \operatorname{Hom}_R(M,N)_{p+q}$:

$$\begin{aligned}(af)_t(rm) &= a(f_t(rm)) = a(rf_t(m)) = (ar)f_t(m)\\ &= (ra)f_t(m) = r(af_t(m)) = r(af)_t(m).\end{aligned}$$

Next, we show that af is A-linear. For this, let $b \in A_s$:

$$\begin{aligned}(af)_{s+t}(bm) &= a(f_{s+t}(bm)) = (-1)^{qs}a(bf_{s+t}(m))\\ &= (-1)^{qs+ps}b(af_{s+t}(m)) = (-1)^{(q+p)s}b((af)_{s+t}(m)).\end{aligned}$$

The first and fourth equalities are by definition of af. The second equality follows because f is A-linear. The third equality is from the graded commutativity and associativity of A. Since $|b| = s$ and $|af| = q+p$, this shows that af is A-linear.

Next, we verify the DG A-module axioms for $\operatorname{Hom}_A(M,N)$. The graded axiom has already been verified. For associativity, continue with the notation from above. We need to show that $a(bf) = (ab)f$, so we compute:

$$(a(bf))_t(m) = a((bf)_t(m)) = a(b(f_t(m))) = (ab)f_t(m) = ((ab)f)_t(m).$$

The third equality is by associativity, and the other equalities are by definition. Distributivity and unitality are verified similarly. Thus, it remains to verify the Leibniz Rule. For this, we need to show that

$$\partial_{p+q}^{\operatorname{Hom}_A(M,N)}(af) = \partial_p^A(a)f + (-1)^p a\partial_q^{\operatorname{Hom}_A(M,N)}(f).$$

For this, we evaluate at m:

$$\begin{aligned}&\partial_{p+q}^{\operatorname{Hom}_A(M,N)}(af)_t(m)\\ &\quad= \partial_{p+q}^N((af)_t(m)) - (-1)^{p+q}(af)_{t-1}(\partial_t^M(m))\\ &\quad= \partial_{t+p+q}^N(a(f_t(m))) - (-1)^{p+q}a(f_{t-1}(\partial_t^M(m)))\\ &\quad= \partial_p^A(a)f_t(m) + (-1)^p a\partial_{t+q}^N(f_t(m))) - (-1)^{p+q}a(f_{t-1}(\partial_t^M(m)))\\ &\quad= \partial_p^A(a)f_t(m) + (-1)^p a[\partial_{t+q}^N(f_t(m))) - (-1)^q(f_{t-1}(\partial_t^M(m)))]\\ &\quad= \partial_p^A(a)f_t(m) + (-1)^p a[\partial_q^{\operatorname{Hom}_A(M,N)}(f)_t(m))]\\ &\quad= (\partial_p^A(a)f)_t(m) + (-1)^p(a\partial_q^{\operatorname{Hom}_A(M,N)}(f))_t(m))]\\ &\quad= \left(\partial_p^A(a)f + (-1)^p a\partial_q^{\operatorname{Hom}_A(M,N)}(f)\right)_t(m)\end{aligned}$$

The third equality is by the Leibniz Rule on N. The fourth step is by distributivity. The remaining equalities are by definition.

(b) Let $a \in A_p$. From the graded axiom for M, we know that the operation $m \mapsto am$ maps $M_t \to M_{p+t}$. The fact that this is R-linear follows from associativity:

$$a(rm) = (ar)m = (ra)m = r(am).$$

To show that it is A-linear, let $b \in A_s$, and compute:

$$\mu^{M,a}_{s+t}(bm) = a(bm) = (-1)^{ps}b(am) = (-1)^{ps}b\mu^{M,a}_t(m).$$

(c) Argue as in the proof of Exercise 3.4 in 10.2. $\square$

10.42 (Sketch of Solution to Exercise 7.7) We prove that $\operatorname{Hom}_A(N,f)$ is a morphism of DG A-modules; the argument for $\operatorname{Hom}_A(f,N)$ is similar. To this end, first note that since f is a chain map of R-complexes, Exercise 3.14 shows that the induced map $\operatorname{Hom}_R(N,f)\colon \operatorname{Hom}_R(N,L) \to \operatorname{Hom}_R(N,M)$ is a chain map of R-complexes. Also, note that $\operatorname{Hom}_R(N,f)$ and $\operatorname{Hom}_A(N,f)$ are given by the same composition-with-f rule.

We need to show that $\operatorname{Hom}_A(f,N)$ is well-defined. For this, let $g = \{g_j\} \in \operatorname{Hom}_A(N,L)_q$. We need to show that $\operatorname{Hom}_A(f,N)(g) \in \operatorname{Hom}_A(N,M)_q$, that is, that $\operatorname{Hom}_A(f,N)(g)$ is A-linear. For this, let $a \in A_p$ and $n \in N_t$. We need to show that $\operatorname{Hom}_A(f,N)(g)_{p+t}(an) = a\operatorname{Hom}_A(f,N)(g)_t(n)$. We compute:

$$\begin{aligned}
\operatorname{Hom}_A(f,N)(g)_{p+t}(an) &= f_{p+q+t}(g_{p+t}(an))\\
&= (-1)^{pq}f_{p+q+t}(ag_t(n))\\
&= (-1)^{pq}a[f_{q+t}(g_t(n))]\\
&= (-1)^{pq}a[\operatorname{Hom}_A(f,N)(g)_t(n)]\\
&= (-1)^{pq}[a\operatorname{Hom}_A(f,N)(g)]_t(n).
\end{aligned}$$

The second equality is by the A-linearity of g. The third equality is by the A-linearity of f. The remaining steps are by definition. since $|\operatorname{Hom}_A(f,N)(g)| = |g| = q$ and $|a| = p$, this is the desired equality.

Next, we need to show that $\operatorname{Hom}_A(N,f)$ respects multiplication on A. For this, we use the same letters as in the previous paragraph. We need to show that

$$\operatorname{Hom}_A(N,f)(ag) = a[\operatorname{Hom}_A(N,f)(g)]$$

so we compute:

$$\begin{aligned}
\operatorname{Hom}_A(N,f)(ag)_t(n) &= f_{t+p+q}((ag)_t(n))\\
&= f_{t+p+q}(a(g_t(n)))\\
&= a(f_{t+q}(g_t(n)))\\
&= a(\operatorname{Hom}_A(N,f)(g)_t(n))\\
&= a[\operatorname{Hom}_A(N,f)(g)]_t(n).
\end{aligned}$$

The third equality is by the A-linearity of f, and the others are by definition. $\square$

10.43 (Sketch of Solution to Exercise 7.9) We begin by showing that $M \otimes_A N$ is an R-complex, using Lemma 10.35(a). Thus, we need to show that U is a subcomplex of $M \otimes_R N$. For this, it suffices to show that the differential $\partial^{M\otimes_R N}$ maps each generator of U into U. To this end, let $a \in A_p$, $m \in M_q$ and $n \in N_s$, and compute:

$$\begin{aligned}
&\partial^{M\otimes_R N}_{p+q+s}((am)\otimes n-(-1)^{pq}m\otimes(an))\\
&\qquad= \partial^M_{p+q}(am)\otimes n+(-1)^{p+q}(am)\otimes\partial^N_s(n)\\
&\qquad\quad -(-1)^{pq}\partial^M_q(m)\otimes(an)-(-1)^{pq+q}m\otimes\partial^N_{p+s}(an)\\
&\qquad= [\partial^A_p(a)m]\otimes n+(-1)^p[a\partial^M_q(m)]\otimes n\\
&\qquad\quad +(-1)^{p+q}(am)\otimes\partial^N_s(n)-(-1)^{pq}\partial^M_q(m)\otimes(an)\\
&\qquad\quad -(-1)^{pq+q}m\otimes[\partial^A_p(a)n]-(-1)^{pq+q+p}m\otimes[a\partial^N_s(n)]\\
&\qquad= [\partial^A_p(a)m]\otimes n-(-1)^{pq+q}m\otimes[\partial^A_p(a)n]\\
&\qquad\quad +(-1)^p[a\partial^M_q(m)]\otimes n-(-1)^{pq}\partial^M_q(m)\otimes(an)\\
&\qquad\quad +(-1)^{p+q}(am)\otimes\partial^N_s(n)-(-1)^{pq+q+p}m\otimes[a\partial^N_s(n)]\\
&\qquad= [\partial^A_p(a)m]\otimes n-(-1)^{(p-1)q}m\otimes[\partial^A_p(a)n]\\
&\qquad\quad +(-1)^p\left[[a\partial^M_q(m)]\otimes n-(-1)^{p(q-1)}\partial^M_q(m)\otimes(an)\right]\\
&\qquad\quad +(-1)^{p+q}\left[(am)\otimes\partial^N_s(n)-(-1)^{pq}m\otimes[a\partial^N_s(n)]\right].
\end{aligned}$$

Since each of these terms is a multiple of a generator of U, we conclude that $\partial^{M\otimes_R N}_{p+q+s}((am)\otimes n-(-1)^{pq}m\otimes(an))\in U$, as desired.

As in the solution to Exercise 5.13 contained in 10.29, the R-complex $M\otimes_R N$ has a well-defined DG A-module structure defined on generators by the formula $b(m\otimes n):=(bm)\otimes n$. To show that the same formula is well-defined on $M\otimes_A N$, we need to show that multiplication by $b\in A_t$ maps each generator of U into U:

$$\begin{aligned}
b((am)\otimes n-(-1)^{pq}m\otimes(an)) &= (b(am))\otimes n-(-1)^{pq}(bm)\otimes(an)\\
&= (-1)^{pt}(a(bm))\otimes n-(-1)^{pq}(bm)\otimes(an)\\
&= (-1)^{pt}\left[(a(bm))\otimes n-(-1)^{p(q+t)}(bm)\otimes(an)\right].
\end{aligned}$$

Since $|bm|=q+t$ and $|a|=p$, this is in U, as desired.

Now that we know that the differential and the scalar multiplication of A on $M\otimes_A N$ are well-defined, the other DG A-module axioms are inherited from $M\otimes_R N$. Finally, the formula $(am)\otimes n=(-1)^{|a||m|}m\otimes(an)$ in $M\otimes_A N$ follows from the condition $(am)\otimes n-(-1)^{|a||m|}m\otimes(an)\in U$. $\square$

10.44 (Sketch of Solution to Exercise 7.10) Let $f\colon A\to B$ be a morphism of DG R-algebras. The DG R-algebra B is a DG A-module via the scalar multiplication $a_ib:=f_i(a_i)b$, by Exercise 5.12(c). So, we know from Exercise 7.9 that $B\otimes_A M$ has a well-defined DG A-module structure. It remains to show that it has a well-defined DG A-module structure by the action $b(b'\otimes m):=(bb')\otimes m$. Notice that, once this is shown, the compatibility with the DG A-module structure is automatic:

$$a_i(b \otimes m) = (a_i b) \otimes m = (f_i(a_i)b) \otimes m = f_i(a_i)(b \otimes m).$$

Let U be the R-submodule of $B \otimes_R M$ generated by all elements of the following form: $(ab) \otimes m - (-1)^{|a||b|} b \otimes (am)$.

We show that the DG B-module structure on $B \otimes_A M$ is well-defined. For this, note that $B \otimes_R M$ has a well-defined DG B-module structure via the composition $R \to A \to B$, by Exercise 5.13. Thus, it suffices to let $c \in B_p$ and show that multiplication by c maps generators of U into U:

$$\begin{aligned}
&c((ab) \otimes m - (-1)^{|a||b|} b \otimes (am)) \\
&\qquad = (c(ab)) \otimes m - (-1)^{|a||b|}(cb) \otimes (am) \\
&\qquad = (-1)^{|c||a|}(a(cb)) \otimes m - (-1)^{|a||b|}(cb) \otimes (am) \\
&\qquad = (-1)^{|c||a|}\Big[(a(cb)) \otimes m - (-1)^{|a|(|b|+|c|)}(cb) \otimes (am)\Big] \\
&\qquad = (-1)^{|c||a|}\Big[(a(cb)) \otimes m - (-1)^{|a|(|cb|)}(cb) \otimes (am)\Big] \in U.
\end{aligned}$$

For the DG B-module axioms, the only one with any substance is the Leibniz Rule:

$$\begin{aligned}
&\partial^{B \otimes_A M}_{|c|+|b|+|m|}(c(b \otimes m)) \\
&\qquad = \partial^{B \otimes_A M}_{|c|+|b|+|m|}((cb) \otimes m) \\
&\qquad = \partial^{B}_{|c|+|b|}(cb) \otimes m + (-1)^{|c|+|b|}(cb) \otimes \partial^{M}_{|m|} m) \\
&\qquad = (\partial^{B}_{|c|}(c)b) \otimes m + (-1)^{|c|}(c\partial^{B}_{|b|}(b)) \otimes m + (-1)^{|c|+|b|}(cb) \otimes \partial^{M}_{|m|} m) \\
&\qquad = \partial^{B}_{|c|}(c)(b \otimes m) + (-1)^{|c|} c\Big[(\partial^{B}_{|b|}(b)) \otimes m + (-1)^{|b|}(b) \otimes \partial^{M}_{|m|} m)\Big] \\
&\qquad = \partial^{B}_{|c|}(c)(b \otimes m) + (-1)^{|c|} c\partial^{B \otimes_A M}_{|c|+|b|+|m|}(b \otimes m).
\end{aligned}$$

The third equality follows from the Leibniz Rule for B, and the fourth equality is by distributivity. The remaining equalities are by definition. □

10.45 (Sketch of Solution to Exercise 7.11)

Hom cancellation. For each $f = \{f_p\} \in \operatorname{Hom}_A(A,L)_i$ we have $f_0 \colon A_0 \to L_i$, hence $f_0(1_A) \in L_i$. Define $\alpha_i \colon \operatorname{Hom}_A(A,L)_i \to L_i$ by the formula $\alpha_i(f) := f_0(1_A)$. We show that $\alpha \colon \operatorname{Hom}_A(A,L) \to L$ is a morphism of DG A-modules and that it is bijective, by Remark 5.21. To show that it is a chain map over R, we compute:

$$\begin{aligned}
\alpha_{i-1}(\partial_i^{\operatorname{Hom}_A(A,L)}(f)) &= \alpha_{i-1}(\{\partial^Y_{p+i} f_p - (-1)^i f_{p-1}\partial^X_p\}) \\
&= \partial^L_i(f_0(1_A)) - (-1)^i f_{-1}\partial^A_0(1_A) \\
&= \partial^L_i(f_0(1_A)) \\
&= \partial^L_i(\alpha_i(f)).
\end{aligned}$$

To show that α is A-linear, let $a \in A_j$ and compute:

$$\alpha_{i+j}(af) = (af)_0(1_A) = af_0(1_A) = a\alpha_i(f).$$

To see that α is injective, suppose that $0 = \alpha_i(f) = f_0(1_A)$. It follows that for all $a \in A_j$ we have

$$f_j(a) = f_j(a1_A) = af_0(1_A) = a \cdot 0 = 0.$$

We conclude that α is injective. To show that α is surjective, let $x \in L_i$. As in the proof of Exercise 7.3(b) in 10.41, the map $\nu^x \colon A \to L$ given by $a \mapsto ax$ is a homomorphism of degree i. Moreover, we have

$$\alpha_i(\nu^x) = \nu_0^x(1_A) = 1_A x = x$$

so α is surjective, as desired.

Notice that the special case $L = A$ explains the isomorphism $\mathrm{Hom}_A(A,A) \cong A$.

Tensor cancellation. Define $\eta \colon A \otimes_R L \to L$ by the formula $\eta_{i+j}(a_i \otimes x_j) := a_i x_j$. This is a well-defined chain map by Exercise 5.15. Let U be the R-submodule of $A \otimes_R L$ generated by the elements of the form $(a_i b_j) \otimes x_k - (-1)^{ij} b_j \otimes (a_i x_k)$. For each such generator, we have

$$\eta_{i+j+k}((a_i b_j) \otimes x_k - (-1)^{ij} b_j \otimes (a_i x_k)) = (a_i b_j)x_k - (-1)^{ij} b_j(a_i x_k) = 0$$

since $b_j a_i = (-1)^{ij} a_i b_j$. It follows that η induces a well-defined map $\nu \colon A \otimes_A L \to L$ given by $\nu_{i+j}(a_i \otimes x_j) := a_i x_j$. Since η is a chain map, it follows readily that ν is also a chain map. Moreover, it is A-linear because

$$\nu_{i+j+k}(a_i(b_j \otimes x_k)) = \nu_{i+j+k}((a_i b_j) \otimes x_k) = (a_i b_j)x_k = a_i(b_j x_k) = a_i \nu_{j+k}(b_j \otimes x_k).$$

To show that ν is an isomorphism, we construct a two-sided inverse. Let $\beta \colon L \to A \otimes_A L$ be given by $\beta_i(x_i) = 1 \otimes x_i$. As in previous exercises, this is a well-defined morphism of DG A-modules. To see that it is a two-sided inverse for ν, we compute:

$$\beta_{i+j}(\nu_{i+j}(a_i \otimes x_j)) = 1 \otimes (a_i x_j) = a_i \otimes x_j.$$

This shows that $\beta\nu$ is the identity on $A \otimes_A L$. The fact that $\nu\beta$ is the identity on L is even easier.

Again, the special case $L = A$ explains the isomorphism $A \otimes_A A \cong A$.

Tensor commutativity. By Exercise 4.2(c), the map $L \otimes_R M \xrightarrow{\gamma} M \otimes_R L$ given by $x_i \otimes y_j \mapsto (-1)^{ij} y_j \otimes x_i$ is a well-defined isomorphism of R-complexes. Let V be the submodule of $L \otimes_R M$ generated over R by the elements of the form $(a_i x_j) \otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k)$. Let W be the R-submodule of $M \otimes_R L$ generated by the elements of the form $(a_i y_j) \otimes x_k - (-1)^{ij} y_j \otimes (a_i x_k)$.

For each element $(a_i x_j) \otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k) \in U$, we have

$$\begin{aligned}
\gamma_{i+j+k}((a_i x_j)\otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k)) \\
&= (-1)^{(i+j)k} y_k \otimes (a_i x_j) - (-1)^{ij+(i+k)j} (a_i y_k) \otimes x_j \\
&= (-1)^{(i+j)k} y_k \otimes (a_i x_j) - (-1)^{jk} (a_i y_k) \otimes x_j \\
&= -(-1)^{jk} [(a_i y_k) \otimes x_j - (-1)^{ik} y_k \otimes (a_i x_j)] \in W.
\end{aligned}$$

It follows that γ factors through the natural epimorphisms $L \otimes_R M \to L \otimes_A M$ and $M \otimes_R L \to M \otimes_A L$, that is, the map $\overline{\gamma}\colon L \otimes_A M \to M \otimes_A L$ given by $x_i \otimes y_j \mapsto (-1)^{ij} y_j \otimes x_i$ is well-defined. To show that $\overline{\gamma}$ is A-linear, we compute:

$$\begin{aligned}
\overline{\gamma}_{i+j+k}(a_i(x_j \otimes y_k)) &= \overline{\gamma}_{i+j+k}((a_i x_j) \otimes y_k) \\
&= (-1)^{(i+j)k} y_k \otimes (a_i x_j) \\
&= (-1)^{(i+j)k+ik} (a_i y_k) \otimes x_j \\
&= (-1)^{jk} a_i (y_k \otimes x_j) \\
&= a_i \overline{\gamma}_{j+k}(x_j \otimes y_k).
\end{aligned}$$

Similarly, the map $\overline{\delta}\colon M \otimes_A L \to L \otimes_A M$ given by $y_j \otimes x_i \mapsto (-1)^{ij} x_i \otimes y_j$ is well-defined and A-linear. It is straightforward to show that the compositions $\overline{\gamma}\overline{\delta}$ and $\overline{\delta}\overline{\gamma}$ are the respective identities, so that $\overline{\gamma}$ is the desired isomorphism. $\square$

10.46 (Sketch of Solution to Exercise 7.15) For the map $N \otimes_A f$, let U be the R-submodule of $N \otimes_R L$ generated by the elements $(a_i x_j) \otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k)$, and let V be the R-submodule of $N \otimes_R M$ generated by the elements of the form $(a_i x_j) \otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k)$. To show that $N \otimes_A f$ is well-defined, it suffices to show that the map $N \otimes_R f\colon N \otimes_R L \to N \otimes_R M$ sends each generator of U into V:

$$\begin{aligned}
(N \otimes_R f)_{i+j+k}((a_i x_j) \otimes y_k - (-1)^{ij} x_j \otimes (a_i y_k)) \\
&= (a_i x_j) \otimes f_k(y_k) - (-1)^{ij} x_j \otimes f_{i+k}(a_i y_k) \\
&= (a_i x_j) \otimes f_k(y_k) - (-1)^{ij} x_j \otimes (a_i f_k(y_k)) \in V.
\end{aligned}$$

To show that $N \otimes_A f$ is A-linear, we compute similarly:

$$\begin{aligned}
(N \otimes_A f)_{i+j+k}(a_i(x_j \otimes y_k)) &= (N \otimes_A f)_{i+j+k}((a_i x_j) \otimes y_k)) \\
&= (a_i x_j) \otimes f_k(y_k) \\
&= a_i (x_j \otimes f_k(y_k)) \\
&= a_i (N \otimes_A f)_{j+k}(x_j \otimes y_k)
\end{aligned}$$

The map $f \otimes_R N$ is treated similarly. $\square$

10.47 (Sketch of Solution to Exercise 7.17) We are working over R as a DG R-algebra, which has $R^\natural = R$. Since R is local, we know that a direct sum $\bigoplus_i M_i$ of R-modules is free if and only if each M_i is free. (In general, the M_i are projective; since R is local, we know that projective implies free.) If L is a semi-free DG R-

module, then it is bounded below by definition, and the module $\bigoplus_i L_i$ is free over R, so each L_i is free, as desired. The converse is handled similarly.

If F is a free resolution of M, then the previous paragraph implies that F is semi-free. Exercise 3.10 implies that there is a quasiisomorphism $F \xrightarrow{\simeq} M$ over R, so this is a semi-free resolution by definition. $\square$

10.48 (Sketch of Solution to Exercise 7.18) It is straightforward to show that M is exact (as an R-complex) if and only if the natural map $0 \to M$ is a quasiisomorphism, since the induced map on homology is the natural map $0 \to \mathrm{H}_i(M)$. Exercise 7.17 implies that 0 is semi-free, so the map $0 \to M$ is a quasiisomorphism if and only if it is a semi-free resolution.

Since A is bounded below, so are $\Sigma^n A$ and $\bigoplus_{n \geqslant n_0} \Sigma^n A^{\beta_n}$. To show that $\Sigma^n A$ is semi-free, we need to show that $1_A \in (\Sigma^n A)_n$ is a semibasis. The only subtlety here is in the signs. If n is odd, then we have

$$(\textstyle\sum_i a_i) * 1_A = \sum_i (-1)^i a_i$$
$$(\textstyle\sum_i (-1)^i a_i) * 1_A = \sum_i a_i.$$

The first of these shows that 1_A is linearly independent: if $(\sum_i a_i) * 1_A = 0$, then $\sum_i (-1)^i a_i = 0$ so $a_i = 0$ for all i, which implies that $\sum_i a_i = 0$. The second of these shows that 1_A spans $A^\natural$ over $A^\natural$: for all $\sum_i a_i \in A^\natural$, we have $\sum_i a_i = (\sum_i (-1)^i a_i) * 1_A \in A^\natural \cdot 1_A$. If n is even, then the relevant formula is $(\sum_i a_i) * 1_A = \sum_i a_i$.

To show that $\Sigma^n A^{\beta_n}$ is semi-free, use the previous paragraph to show that the sequence of standard basis vectors $(1_A, 0, \ldots, 0), (0, 1_A, \ldots), \ldots, (0, 0, \ldots, 1_A)$ form a semibasis. To show that $\bigoplus_{n \geqslant n_0} \Sigma^n A^{\beta_n}$ is semi-free, let E_n be a semibasis for each $\Sigma^n A^{\beta_n}$ and show that $\cup_n E_n$ is a semibasis for $\bigoplus_{n \geqslant n_0} \Sigma^n A^{\beta_n}$. $\square$

10.49 (Sketch of Solution to Exercise 7.19)

(a) Exercise 7.10 shows that $K \otimes_R F$ is a DG K-module, so we only need to show that it is semi-free. For each $i \in \mathbb{Z}$, let E_i be a basis of the free R-module F_i, and set $E'_i = \{1_K \otimes e \in K \otimes_R F \mid e \in E_i\}$. We claim that $E' := \cup_i E'_i$ is a semibasis for $K \otimes_R F$. To show that E' spans $(K \otimes_R F)^\natural$, it suffices to show that for each $x \in K_i$ and each $y \in F_j$ the generator $x \otimes y \in (K \otimes_R F)_{i+j}$ is in the K-span of E'. For this, write $y = \sum_{e \in E_j} r_e e$, and compute:

$$x \otimes y = x \otimes (\textstyle\sum_{e \in E_j} r_e e) = \sum_{e \in E_j} r_e x (1_K \otimes e) \in K \cdot E'.$$

To show that E' is linearly independent takes a bit of bookkeeping. Suppose that

$$0 = \sum_{i=1}^{m} x_i (1_K \otimes e_i) \tag{10.49.1}$$

in $K \otimes_A F$ for some $x_i \in K^\natural$ and distinct elements $e_1, \ldots, e_m \in E$. Since $(K \otimes_R F)^\natural$ is a graded $K^\natural$-module, we may assume without loss of generality that each x_i is homogeneous and that the degree $|x_i \otimes e_i| = |x_i| + |e_i| = n$ is the same for all i. Moreover, $(K \otimes_R F)^\natural$ is a bi-graded $K^\natural$-module (with gradings coming from K and

F) so we may assume without loss of generality that the degree $|x_i| = p$ is the same for all i and that the degree $|e_i| = q$ is the same for all i. Thus, equation (10.49.1) becomes $0 = \sum_{i=1}^{m} x_i \otimes e_i$. This sum occurs in the following submodule of $K_p \otimes_R F_q$:

$$\bigoplus_{i=1}^{m} K_p \otimes_R Re_i \cong \bigoplus_{i=1}^{m} K_p.$$

Under this isomorphism, the element $0 = \sum_{i=1}^{m} x_i \otimes e_i$ corresponds to the vector $0 = (x_1, \ldots, x_m)$ which implies that $x_i = 0$ for $i = 1, \ldots, m$. It follows that E' is linearly independent, as desired.

(b) Let $F \xrightarrow{\simeq} M$ be a semi-free resolution of a DG R-module M. Part (a) implies that $K \otimes_R F$ is semi-free over K. Since K is a bounded below complex of projective R-modules, Fact 4.9 implies that the induced map $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a quasiisomoprhism, so it is a semi-free resolution by definition. $\square$

10.50 (Sketch of Solution to Exercise 7.26) To show that $\chi_M^A \colon A \to \mathrm{Hom}_A(M,M)$ has the desired properties, we first note that Exercise 7.3(b) shows that it maps A_i to $\mathrm{Hom}_A(M,M)_i$ for all i. Next, we check that χ_M^A is a chain map:

$$\begin{aligned}
\partial_i^{\mathrm{Hom}_R(M,M)}((\chi_M^A)_i(a_i)) &= \{\partial_{i+p}^M \mu_p^{M,a_i} - (-1)^i \mu_{p-1}^{M,a_i} \partial_p^M\} \\
(\chi_M^A)_{i-1}(\partial_i^A(a_i)) &= \{\mu_p^{M,\partial_i^A(a_i)}\}.
\end{aligned}$$

To see that these are equal, we evaluate at $m_p \in M_p$:

$$\begin{aligned}
\partial_{i+p}^M(\mu_p^{M,a_i}(m_p)) - (-1)^i \mu_{p-1}^{M,a_i}(\partial_p^M(m_p)) \\
&= \partial_{i+p}^M(a_i m_p) - (-1)^i a_i \partial_p^M(m_p) \\
&= \partial_i^A(a_i) m_p + (-1)^i a_i \partial_p^M(m_p) - (-1)^i a_i \partial_p^M(m_p) \\
&= \partial_i^A(a_i) m_p \\
&= \mu_p^{M,\partial_i^A(a_i)}(m_p)
\end{aligned}$$

To complete the proof, we check that χ_M^A is a A-linear. For this, we need to show that $(\chi_M^A)_{i+j}(a_i b_j) = a_i (\chi_M^A)_j(b_j)$. To show this, we evaluate at $m_p \in M_p$:

$$(\chi_M^A)_{i+j}(a_i b_j)_p(m_p) = a_i b_j m_p = a_i (\chi_M^A)_j(b_j)_p(m_p)$$

as desired. $\square$

10.51 (Sketch of Solution to Exercise 7.32) Let M and N be R-modules. Exercise 7.17 implies that each free resolution F of M gives rise to a semi-free resolution $F \xrightarrow{\simeq} M$. Thus, the module $\mathrm{Ext}_R^i(M,N)$ defined in 7.31 is $\mathrm{H}_{-i}(\mathrm{Hom}_R(F,M))$, which is the usual $\mathrm{Ext}_R^i(M,N)$. $\square$

10.52 (Sketch of Solution to Exercise 8.4) We begin with the graded vector space $W'' = 0 \bigoplus Fw_2 \bigoplus Fw_1 \bigoplus Fw_0 \bigoplus 0$. The differential ∂'' consists of two matrices of size 1×1:

$$0 \to Fw_2 \xrightarrow{x_2} Fw_1 \xrightarrow{x_1} Fw_0 \to 0.$$

The condition $\partial''_{i-1}\partial''_i = 0$ is only non-trivial for $i = 2$, in which case it boils down to the following:

$$0 = \partial''_1(\partial''_2(w_2)) = \partial''_1(x_2 w_1) = x_1 x_2 w_0.$$

We conclude that (W'', ∂'') is an R-complex if and only if

$$x_1 x_2 = 0. \tag{10.52.1}$$

The scalar multiplication of U on W'' is completely described by specifying ew_0 and ew_1, and this requires two more elements $y_0, y_1 \in F$ so that we have $ew_0 = y_0 w_1$ and $ew_1 = y_1 w_2$. The associative law (which was not a concern for W and W') says that we must have

$$0 = 0w_0 = e^2 w_0 = e(ew_0) = e(y_0 w_1) = y_0 y_1 w_2$$

so we conclude that

$$y_0 y_1 = 0. \tag{10.52.2}$$

Note that once this is satisfied, the general associative law follows. This leaves the Leibniz Rule for the products ew_0, ew_1, and ew_2. We begin with ew_0:

$$\begin{aligned}
\partial''_1(ew_0) &= \partial''^U_1(e)w_0 + (-1)^{|e|} e\partial''_0(w_0) \\
\partial''_1(y_0 w_1) &= 0w_0 + (-1)^{|e|} e0 \\
x_1 y_0 w_0 &= 0
\end{aligned}$$

which implies that

$$x_1 y_0 = 0. \tag{10.52.3}$$

A similar computation for ew_2 shows that

$$x_2 y_1 = 0. \tag{10.52.4}$$

A last computation for ew_1 yields $x_2 y_1 + x_1 y_0 = 0$, which is redundant because of equations (10.52.3)–(10.52.4). Thus, $\operatorname{Mod}^U(W'')$ consists of all ordered quadruplets $(x_1, x_2, y_0, y_1) \in \mathbb{A}^4_F$ satisfying the equations (10.52.1)–(10.52.4). It is possibly worth noting that the ideal defined by equations (10.52.1)–(10.52.4) has a simple primary decomposition:

$$(x_1 x_2, y_0 y_1, x_1 y_0, x_2 y_1) = (x_1 y_1) \cap (x_2, y_0).$$

Next, we repeat this process for $W''' = 0 \bigoplus Fz_2 \bigoplus (Fz_{1,1} \bigoplus Fz_{1,2}) \bigoplus Fz_0 \bigoplus 0$. The differential ∂''' in this case has the following form:

$$0 \to Fz_2 \xrightarrow{\left(\begin{smallmatrix} a_{2,1} \\ a_{2,2} \end{smallmatrix}\right)} Fz_{1,1} \bigoplus Fz_{1,2} \xrightarrow{(a_{1,1}\ a_{1,2})} Fz_0 \to 0$$

meaning that $\partial_2'''(z_2) = a_{2,1}z_{1,1} + a_{2,2}z_{1,2}$ and $\partial_1'''(z_{1,i}) = a_{1,i}z_0$ for $i = 1,2$. Scalar multiplication also requires more letters:

$$\begin{aligned} ez_0 &= b_{0,1}z_{1,1} + b_{0,2}z_{1,2} \\ ez_{1,1} &= b_{1,1}z_2 \\ ez_{1,2} &= b_{1,2}z_2. \end{aligned}$$

The condition $\partial_{i-1}'''\partial_i''' = 0$ is equivalent to the following equation:

$$a_{1,1}a_{2,1} + a_{1,2}a_{2,2} = 0. \tag{10.52.5}$$

The associative law is equivalent to the next equation:

$$b_{0,1}b_{1,1} + b_{0,2}b_{1,2} = 0. \tag{10.52.6}$$

For the Leibniz Rule, we need to consider the products ez_0, $ez_{1,j}$ and ez_2, so this axiom is equivalent to the following equations:

$$a_{1,1}b_{0,1} + a_{1,2}b_{0,2} = 0 \tag{10.52.7}$$
$$a_{2,i}b_{1,j} + a_{1,j}b_{0,i} = 0 \qquad \text{for all } i = 1,2 \text{ and } j = 1,2 \tag{10.52.8}$$
$$a_{2,1}b_{1,1} + a_{2,2}b_{1,2} = 0. \tag{10.52.9}$$

So, $\mathrm{Mod}^U(W'')$ consists of all $(a_{1,1}, a_{1,2}, a_{2,1}, a_{2,2}, b_{0,1}, b_{0,2}, b_{1,1}, b_{1,2}) \in \mathbb{A}_F^8$ satisfying the equations (10.52.5)–(10.52.9). $\square$

10.53 (Sketch of Solution to Exercise 8.7) We recall that

$$\begin{aligned} W'' = &\quad 0 \bigoplus Fw_2 \bigoplus Fw_1 \bigoplus Fw_0 \bigoplus 0 \\ W''' = &\quad 0 \bigoplus Fz_2 \bigoplus (Fz_{1,1} \bigoplus Fz_{1,2}) \bigoplus Fz_0 \bigoplus 0. \end{aligned}$$

It follows that

$$\begin{aligned} \mathrm{End}_F(W'')_0 &= \bigoplus_{i=0}^{2} \mathrm{Hom}_F(Fw_i, Fw_i) \cong F^3 = \mathbb{A}_F^3 \\ \mathrm{GL}_F(W'')_0 &= \bigoplus_{i=0}^{2} \mathrm{Aut}_F(Fw_i) \cong (F^\times)^3 = U_{u_2u_1u_0} \subset \mathbb{A}_F^3 \\ \mathrm{End}_F(W''')_0 &\cong \mathrm{Hom}_F(F,F) \bigoplus \mathrm{Hom}_F(F^2,F^2) \bigoplus \mathrm{Hom}_F(F,F) \\ &\cong F \times F^4 \times F = \mathbb{A}_F^6 \\ \mathrm{GL}_F(W''')_0 &= \mathrm{Aut}_F(F) \bigoplus \mathrm{Aut}_F(F^2) \bigoplus \mathrm{Aut}_F(F) \\ &\cong F^\times \times \mathrm{GL}_2(F) \times F^\times = U_{c_2(c_{11}c_{22} - c_{12}c_{21})c_0} \subset \mathbb{A}_F^6. \qquad \square \end{aligned}$$

10.54 (Sketch of Solution to Exercise 8.10) We continue with the notation of Example 5.17 and the solutions to Exercises 8.4 and 8.7. Under the isomorphism

$\mathrm{GL}_F(W'')_0 \cong U_{u_2u_1u_0} \subseteq \mathbb{A}^3_F$, an ordered triple $(u_0,u_1,u_2) \in U_{u_2u_1u_0}$ corresponds to the isomorphism

$$\begin{array}{ccccccccc}
\partial: & 0 & \longrightarrow & Fw_2 & \xrightarrow{x_2} & Fw_1 & \xrightarrow{x_1} & Fw_0 & \longrightarrow 0 \\
\alpha \downarrow \cong & & & \downarrow u_2 & & \downarrow u_1 & & \downarrow u_0 & \\
\widetilde{\partial}: & 0 & \longrightarrow & F\widetilde{w}_2 & \xrightarrow{\widetilde{x}_2} & F\widetilde{w}_1 & \xrightarrow{\widetilde{x}_1} & F\widetilde{w}_0 & \longrightarrow 0.
\end{array}$$

Let $e \cdot_\alpha \widetilde{w}_j = \widetilde{y}_j \widetilde{w}_{j+1}$ for $j = 0,1$. Then direct computations as in Example 8.9 show that $\widetilde{x}_i = u_{i-1}x_iu_i^{-1}$ for $i = 1,2$ and $\widetilde{y}_j = u_{j+1}y_ju_j^{-1}$ for $j = 0,1$.

Under the isomorphism $\mathrm{GL}_F(W''')_0 \cong U_{c_2(c_{11}c_{22}-c_{12}c_{21})c_0} \subset \mathbb{A}^6_F$, an ordered sextuple $(c_2,c_{11},c_{22},c_{12},c_{21},c_0) \in U_{c_2(c_{11}c_{22}-c_{12}c_{21})c_0}$ corresponds to the isomorphism

$$\begin{array}{ccccccccc}
\partial: & 0 & \longrightarrow & Fz_2 & \xrightarrow{\left(\begin{smallmatrix} a_{2,1} \\ a_{2,2}\end{smallmatrix}\right)} & Fz_{1,1} \oplus Fz_{1,2} & \xrightarrow{(a_{1,1}\ a_{1,2})} & Fz_0 & \longrightarrow 0 \\
\alpha \downarrow \cong & & & \downarrow c_2 & & \downarrow \left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{smallmatrix}\right) & & \downarrow c_0 & \\
\widetilde{\partial}: & 0 & \longrightarrow & F\widetilde{z}_2 & \xrightarrow[\left(\begin{smallmatrix} \widetilde{a}_{2,1} \\ \widetilde{a}_{2,2}\end{smallmatrix}\right)]{} & F\widetilde{z}_{1,1} \oplus F\widetilde{z}_{1,2} & \xrightarrow[(\widetilde{a}_{1,1}\ \widetilde{a}_{1,2})]{} & F\widetilde{z}_0 & \longrightarrow 0.
\end{array}$$

Set $\Delta = \det\left(\begin{smallmatrix} c_{11} & c_{12} \\ c_{21} & c_{22}\end{smallmatrix}\right) = c_{11}c_{22} - c_{12}c_{21}$. Thus, we have the following:

$$\begin{aligned}
\begin{pmatrix} \widetilde{a}_{2,1} \\ \widetilde{a}_{2,2} \end{pmatrix} &= \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \begin{pmatrix} \widetilde{a}_{2,1} \\ \widetilde{a}_{2,2} \end{pmatrix} (c_2^{-1}) = \begin{pmatrix} (c_{11}a_{2,1} + c_{12}a_{2,2})c_2^{-1} \\ (c_{21}a_{2,1} + c_{22}a_{2,2})c_2^{-1} \end{pmatrix} \\
\left(\widetilde{a}_{1,1}\ \widetilde{a}_{1,2}\right) &= (c_0) \begin{pmatrix} a_{2,1} \\ a_{2,2} \end{pmatrix} \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix}^{-1} \\
&= \left(\Delta^{-1}c_0(a_{1,1}c_{22} - a_{1,2}c_{21})\ \ \Delta^{-1}c_0(-a_{1,1}c_{12} + a_{1,2}c_{11})\right).
\end{aligned}$$

In other words, we have

$$\begin{aligned}
\widetilde{a}_{2,i} &= (c_{i1}a_{2,1} + c_{i2}a_{2,2})c_2^{-1} \\
\widetilde{a}_{1,1} &= \Delta^{-1}c_0(a_{1,1}c_{22} - a_{1,2}c_{21}) \\
\widetilde{a}_{1,2} &= \Delta^{-1}c_0(-a_{1,1}c_{12} + a_{1,2}c_{11})
\end{aligned}$$

for $i = 1,2$. For the scalar multiplication, we have

$$e \cdot_\alpha \widetilde{z}_2 = 0$$

and using the rule $e \cdot_\alpha \widetilde{z}_* = \alpha_2(e\alpha_1^{-1}(z_*))$, we find that

$$e\cdot_\alpha \widetilde{z}_{1,1} = c_2\Delta^{-1}(c_{22}b_{1,1}-c_{21}b_{1,2})\widetilde{z}_2$$
$$e\cdot_\alpha \widetilde{z}_{1,2} = c_2\Delta^{-1}(-c_{12}b_{1,1}+c_{11}b_{1,2})\widetilde{z}_2$$
$$e\cdot_\alpha \widetilde{z}_0 = c_0^{-1}(c_{11}b_{0,1}+c_{12}b_{0,2})\widetilde{z}_{1,1}+c_0^{-1}(c_{21}b_{0,1}+c_{22}b_{0,2})\widetilde{z}_{1,2}.$$

In other words, we have

$$\widetilde{b}_{1,1} = c_2\Delta^{-1}(c_{22}b_{1,1}-c_{21}b_{1,2})$$
$$\widetilde{b}_{1,2} = c_2\Delta^{-1}(-c_{12}b_{1,1}+c_{11}b_{1,2})\widetilde{z}_2$$
$$\widetilde{b}_{0,i} = c_0^{-1}(c_{i1}b_{0,1}+c_{i2}b_{0,2})$$

for $i = 1,2$. $\square$

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