KOSZUL COMPLEXES

Let R be a commutative noetherian ring with identity. Although not all work included herein requires R to be noetherian, it simplifies matters quite a bit. This compilation covers the equivalent definitions of the Koszul complex, showing that the Koszul complex is independent of its generating sequence, showing that the Koszul complex is self-dual, depth sensitivity, and the differential graded algebra structure on the Koszul complex.

Additional and supporting information about Koszul complexes can be found in several texts: Bruns and Herzog [2, Section 1.6], Sather-Wagstaff's Homological Algebra Notes [8, Section VI] (these notes can be found online at Sather-Wagstaff's website), Matsumura [5, Section 6], Foxby [4] (these notes are unpublished, see Sather-Wagstaff for information), and Eisenbud [3, Chapter 17].

1. Day 1

In this section we begin by setting notation and defining the Koszul complex. There are three different ways of defining the Koszul complex; we may use the mapping cone, the wedge product, or tensor products in our defintion. We use a different notation for each definition ($K(\underline{x})$, $T(\underline{x})$, and $L(\underline{x})$). We will show the three definitions are equivalent. It should be noted that we can discuss the Koszul complex in terms of 'definition' or 'construction'. Different texts use a different label: for the purposes of these notes we will use the term 'definition'.

Notation 1.1. Let $x_1, x_2, \cdots, x_n \in R$ where $n \in \mathbb{N}$. Set $\underline{x} = x_1, x_2, \cdots, x_n$ and $(\underline{x}') = x_1, x_2, \cdots, x_{n-1}$.

The first definition of the Koszul complex that we present relies on the mapping cone. For background information pertaining to the mapping cone see [8, Chapter VI.3].

Definition 1.2. Let X be an R-complex. For each $r \in R$, the map $\mu^r : X \to X$ defined by $\mu^r_i(m) = rm$ is a chain map. For each $i \in \mathbb{Z}$, the induced map $H_i(\mu^r) : H_i(X) \to H_i(X)$ is given by $\overline{m} \mapsto r\overline{m}$.

Definition 1.3. We define the Koszul complex by induction on n.

Base case: $n = 1$. Then we define the Koszul complex to be the following:

$$K(x_1) = 0 \to R \xrightarrow{x_1} R \to 0.$$

Inductive step: Assume that $n \geqslant 2$ and $K(\underline{x}')$ is defined. Then we define $K(\underline{x})$ as

$$K(\underline{x}) = \mathrm{Cone}(K(\underline{x}') \xrightarrow{x_n} K(\underline{x}')) = \mathrm{Cone}(\mu^{x_n}_{K(\underline{x}')}).$$

Remark 1.4. Note that $\mu^{x_n}_{K(\underline{x}')}$ is a chain map. When we use the definition of the mapping cone, $\mathrm{Cone}(\mu^{x_n}_{K(\underline{x}')})$ is a complex; see [8]. For more information about mapping cones, see [4, 1.24] and [3, Section 17.3].

Notation 1.5. For an R-complex X, we let X_i denote the ith component of X.

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Definition 1.6. Let X be an R-complex. The *shift* of X, denoted ΣX, is defined to be $(\Sigma X)_i = X_{i-1}$ and $\partial_i^{\Sigma X} = -\partial_{i-1}^X$; see [8] for examples.

Fact 1.7. Consider the following short exact sequence:

$$0 \to K(\underline{x}') \to \operatorname{Cone}(\mu^{x_n}_{K(\underline{x}')}) \to \Sigma K(\underline{x}') \to 0$$

where $\operatorname{Cone}(\mu^{x_n}_{K(\underline{x}')}) = K(\underline{x})$. This induces a long exact sequence in homology:

$$\cdots \to H_i(K(\underline{x}')) \to H_i(K(\underline{x})) \to H_{i-1}(K(\underline{x}')) \xrightarrow{\eth_{i-1}=x_n} H_{i-1}(K(\underline{x}')) \to \cdots .$$

Recall that with the mapping cone definition of the Koszul complex we can write:

$$K_i(\underline{x}) = K_i(\underline{x}') \oplus K_{i-1}(\underline{x}')$$
$$K_{i-1}(\underline{x}) = K_{i-1}(\underline{x}') \oplus K_{i-2}(\underline{x}').$$

From the short exact sequence of complexes

$$0 \to K(\underline{x}') \to K(\underline{x}) \to \Sigma K(\underline{x}') \to 0$$

we have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & K_i(\underline{x}') & \longrightarrow & \begin{matrix} K_i(\underline{x}') \\ \oplus \\ K_{i-1}(\underline{x}') \end{matrix} & \xrightarrow{\psi} & K_{i-1}(\underline{x}') & \longrightarrow & 0 \\
 & & \downarrow & & \Big\downarrow \begin{pmatrix} \partial_i^{K(\underline{x}')} & x_n \\ 0 & -\partial_{i-1}^{K(\underline{x}')} \end{pmatrix} & & \Big\downarrow -\partial_{i-1}^{K(\underline{x}')} & & \\
0 & \longrightarrow & K_{i-1}(\underline{x}') & \xrightarrow{\phi} & \begin{matrix} K_{i-1}(\underline{x}') \\ \oplus \\ K_{i-2}(\underline{x}') \end{matrix} & \longrightarrow & K_{i-2}(\underline{x}') & \longrightarrow & 0
\end{array}$$

where ϕ and ψ are the natural injection and surjection maps (respectively). Let $y \in K_{i-1}(\underline{x}')$ such that $y \in \operatorname{Ker}(-\partial_{i-1}^{K(\underline{x}')})$. Then we have the following:

$$\psi(\begin{pmatrix} 0 \\ y \end{pmatrix}) = y$$

$$\begin{pmatrix} \partial_i^{K(\underline{x}')} & x_n \\ 0 & -\partial_{i-1}^{K(\underline{x}')} \end{pmatrix} (\begin{pmatrix} 0 \\ y \end{pmatrix}) = \begin{pmatrix} yx_n \\ -\partial_{i-1}^{K(\underline{x}')}(y) \end{pmatrix} = \begin{pmatrix} yx_n \\ 0 \end{pmatrix}$$

$$\phi(yx_n) = \begin{pmatrix} yx_n \\ 0 \end{pmatrix}$$

We then define $\eth(\overline{y}) = \overline{y}x_n = H_{i-1}(x_n)(\overline{y})$.

Next we define the Koszul complex using tensor products.

Definition 1.8. We define the Koszul complex by induction on n.

Base case: $n = 1$. Then we define the $T(\underline{x})$ to be the following:

$$T(x_1) = 0 \to R \xrightarrow{x_1} R \to 0.$$

Inductive step: Assume $n \geqslant 2$ and $T(\underline{x}')$ is defined. Then we define $T(\underline{x})$ as

$$T(\underline{x}) = T(x_1) \otimes_R T(x_2) \otimes_R \cdots \otimes_R T(x_n).$$

Proposition 1.9. *If X is a chain complex and $y \in R$, then*

$$K(y) \otimes_R X = \operatorname{Cone}(X \xrightarrow{y} X).$$

Proof. We define the map

$$\partial_i^{K(y)\otimes X} : \textstyle\coprod_{p+q=i} K_p(y) \otimes_R X_q \to \coprod_{p+q=i-1} K_p(y) \otimes_R X_q$$

by $r_p \otimes x_q \mapsto \partial_p^{K(y)}(r_p) \otimes x_q + (-1)^p r_p \otimes \partial_q^X(x_q)$. The complex $K(y)$ is defined to be the sequence

$$K(y) = 0 \to R_1 \xrightarrow{y} R_0 \to 0.$$

In degree i, we have:

$$\begin{matrix} X_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\Phi} \begin{matrix} X_{i-1} \\ \oplus \\ X_{i-2} \end{matrix}$$

where

$$\begin{aligned} X_i &\cong R_0 \otimes_R X_i \\ X_{i-1} &\cong R_1 \otimes_R X_{i-1} \\ X_{i-2} &\cong R_1 \otimes_R X_{i-2} \\ \Phi &= \begin{pmatrix} \partial_i^X & y \\ 0 & -\partial_{i-1}^X \end{pmatrix} \end{aligned}$$

and Φ is from the definition of the mapping cone. Thus in degree i we can write

$$\begin{matrix} R_0 \otimes_R X_i \\ \oplus \\ R_1 \otimes_R X_{i-1} \end{matrix} \xrightarrow{\Phi} \begin{matrix} R_0 \otimes_R X_{i-1} \\ \oplus \\ R_1 \otimes_R X_{i-2} \end{matrix} \qquad \text{and} \qquad \begin{matrix} R_0 \otimes_R X_i \\ \oplus \\ R_1 \otimes_R X_{i-1} \end{matrix} \xrightarrow{\partial_i^{K(y)\otimes_R X}} \begin{matrix} R_0 \otimes_R X_{i-1} \\ \oplus \\ R_1 \otimes_R X_{i-2}. \end{matrix}$$

For example, if e is a basis element of R_1 we have the following equalities:

$$\begin{aligned} \Phi \begin{pmatrix} 1 \otimes x_i \\ 0 \end{pmatrix} &= \partial_i^{K(y)\otimes X} \begin{pmatrix} 1 \otimes x_i \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} R_0 \otimes \partial_i^X & y \otimes X_{i-1} \\ 0 & -R_1 \otimes \partial_{i-1}^X \end{pmatrix} \begin{pmatrix} 1 \otimes x_i \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 \otimes \partial_i^X(x_i) \\ 0 \end{pmatrix} \\ \partial_i^{K(y)\otimes X} \begin{pmatrix} 0 \\ e \otimes x_{i-1} \end{pmatrix} &= \begin{pmatrix} R_0 \otimes \partial_i^X & y \otimes X_{i-1} \\ 0 & -R_1 \otimes \partial_{i-1}^X \end{pmatrix} \begin{pmatrix} 0 \\ e \otimes x_{i-1} \end{pmatrix} \\ &= \begin{pmatrix} y \otimes x_{i-1} \\ (-1)e \otimes \partial_{i-1}^X(x_{i-1}) \end{pmatrix}. \end{aligned}$$

Hence we can rewrite Φ as $\Phi = \begin{pmatrix} id_{R_0} \otimes \partial_i^X & y \otimes id_{x_{i-1}} \\ 0 & (-1) id_{R_1} \otimes \partial_{i-1}^X \end{pmatrix}$ □

Proposition 1.10. *The mapping cone definition and the tensor product definition of the Koszul complex are equivalent, that is $T(\underline{x}) = K(\underline{x})$.*

Proof. We proceed by induction on n.

Base case: $n = 1$. By definition $T(x_1) = K(x_1)$.

Inductive step: Assume $n \geqslant 2$ and $T(\underline{x}') = K(\underline{x}')$. Then we have the sequence:

$$\begin{aligned} T(\underline{x}) &= T(\underline{x}') \otimes_R T(x_n) \\ &\cong K(\underline{x}') \otimes_R T(x_n) \\ &\cong K(\underline{x}') \otimes_R K(x_n) \\ &\cong \operatorname{Cone}(\mu^{x_n}_{K(\underline{x}')}) \\ &\cong K(\underline{x}) \end{aligned}$$

where the second step follows from the inductive hypothesis, the third step follows from the base case, and the fourth step follows from Proposition 1.9. □

2. Day 2

Proposition 2.1. *For each integer i, one has $K_i(\underline{x}) \cong R^{\binom{n}{i}}$.*

Proof. We proceed by induction on n.

Base case: $n = 1$. The Koszul complex is the following sequence:

$$K(x_1) = 0 \to R \xrightarrow{x_1} R \to 0.$$

We see from this sequence that each component of the Koszul complex is isomorphic to $R^{\binom{n}{i}}$ for $i = 0, 1$

$$K_0(x_1) = R \cong R^{\binom{1}{0}}$$
$$K_1(x_1) = R \cong R^{\binom{1}{1}}.$$

Inductive step: Assume that the result holds for $K_i(\underline{x}')$. Applying the mapping cone definition of the Koszul complex we have the following sequence:

$$\begin{aligned} K_i(\underline{x}) &= K_{i-1}(\underline{x}') \oplus K_i(\underline{x}') \\ &= R^{\binom{n-1}{i}} \oplus R^{\binom{n-1}{i-1}} \\ &= R^{\binom{n}{i}} \end{aligned}$$

yielding the desired result. □

We now present the third and final definition of the Koszul complex. This definition uses the wedge product. For more information about the wedge product see [2, Section 1.6].

Definition 2.2. Set $L_0(\underline{x}) = R$ with basis element $\{1\}$ and set $L_1(\underline{x}) = R^n$ with the basis elements given by the formal symbols $\{e_1, \cdots, e_n\}$. Set $L_i(\underline{x}) = R^{\binom{n}{i}}$ with the basis elements given by the formal symbols

$$\{e_{j_1} \wedge \cdots \wedge e_{j_i} \mid 1 \leqslant j_1 < \cdots < j_i \leqslant n\}.$$

We define the Koszul complex to be the sequence

$$L(\underline{x}) = 0 \to L_n(\underline{x}) \xrightarrow{\partial_n^{L(\underline{x})}} L_{n-1}(\underline{x}) \xrightarrow{\partial_{n-1}^{L(\underline{x})}} \cdots \xrightarrow{\partial_2^{L(\underline{x})}} L_1(\underline{x}) \xrightarrow{\partial_1^{L(\underline{x})}} L_0(\underline{x}) \to 0$$

with the maps defined as follows:

$$\partial_i^{L(\underline{x})} : L_i(\underline{x}) \to L_{i-1}(\underline{x})$$

where $\partial_i^{L(\underline{x})}(e_{j_1} \wedge \cdots \wedge e_{j_i}) = \sum_{s=1}^{i} (-1)^{s+1} x_{j_s} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i}$. (By $\widehat{e}_{j_s}$ we indicate that e_{j_s} is to be omitted from the wedge product.)

Remark 2.3. Since $L_i(\underline{x})$ is a free module, to define the module homomorphism $\partial_i^{L(\underline{x})}$ we need only to define what happens to the basis vectors. For instance, the map $\partial_1^{L(\underline{x})} : L_1(\underline{x}) \to L_2(\underline{x})$ is defined by mapping $e_j \mapsto x_j$ for $j = 1, \dots, n$.

Example 2.4. Consider the following sequence:

$$0 \to L_3(\underline{x}) \xrightarrow{\partial_3^{L(\underline{x})}} L_2(\underline{x}) \xrightarrow{\partial_2^{L(\underline{x})}} L_1(\underline{x}) \xrightarrow{\partial_1^{L(\underline{x})}} L_0(\underline{x}) \to 0. \tag{2.4.1}$$

By definition, the module $L_0(\underline{x}) \cong R$ with basis $\{1\}$. For the remaining modules we have the following:

$$\begin{aligned} L_1(\underline{x}) &\cong R^3 \text{ with basis } \{e_1, e_2, e_3\}, \\ L_2(\underline{x}) &\cong R^3 \text{ with basis } \{e_1 \wedge e_2, e_1 \wedge e_3, e_2 \wedge e_3\}, \\ L_3(\underline{x}) &\cong R \quad \text{with basis } \{e_1 \wedge e_2 \wedge e_3\}. \end{aligned}$$

To show that the sequence (2.4.1) is exact, we show that the composition of two maps is zero:

$$\begin{aligned} \partial_1^{L(\underline{x})}(\partial_2^{L(\underline{x})}(e_1 \wedge e_3)) &= \partial_1^{L(\underline{x})}(x_1 e_3 - x_3 e_1) \\ &= x_1 \partial_1^{L(\underline{x})}(e_3) - x_3 \partial_1^{L(\underline{x})}(e_1) \\ &= x_1 x_3 - x_3 x_1 \\ &= 0. \end{aligned}$$

It is standard to check that the other basis vectors map to zero. For the other composition map we have the following:

$$\begin{aligned} \partial_2^{L(\underline{x})}(\partial_3^{L(\underline{x})}(e_1 \wedge e_2 \wedge e_3)) &= \partial_2^{L(\underline{x})}(x_1 e_2 \wedge e_3 - x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2) \\ &= x_1 \partial_2^{L(\underline{x})}(e_2 \wedge e_3) - x_2 \partial_2^{L(\underline{x})}(e_1 \wedge e_3) + x_3 \partial_2^{L(\underline{x})}(e_1 \wedge e_2) \\ &= x_1(x_2 e_3 - x_3 e_2) - x_2(x_1 e_3 - x_3 e_1) + x_3(x_1 e_2 - x_2 e_1) \\ &= 0. \end{aligned}$$

The second step follows from the fact that $\partial_2^{L(\underline{x})}$ is an R-module homomorphism, and the fourth step is easily checked. The first and third steps are by definition of the maps. Note that the signs are dependent upon the place an element has in the list and not the element itself.

We can now see the shapes of the maps and write them as matrices:

$$\partial_3^{L(\underline{x})} = \begin{pmatrix} x_3 \\ -x_2 \\ x_1 \end{pmatrix} \qquad \partial_2^{L(\underline{x})} = \begin{pmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{pmatrix} \qquad \partial_1^{L(\underline{x})} = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix}$$

It is standard to show that composition of these matrices yield the zero matrix. This is sufficient to show exactness.

Next we show that Definition 2.2 is consistent with Definitions 1.3 and 1.8 for the Koszul complex. To accomplish this task we need to show three facts: $L_i(\underline{x}) \cong R^{\binom{n}{i}}$, $L(\underline{x})$ is an R-complex, and $L(\underline{x}) \cong K(\underline{x})$.

The first of these three is evident. We will concentrate on showing the second fact for now and will prove the third fact later.

Lemma 2.5. *$L(\underline{x})$ is an R-complex.*

Proof. For any n and any i, we show that $\partial_{i-1}^{L(\underline{x})} \circ \partial_i^{L(\underline{x})} = 0$. Let $e_{j_1} \wedge \cdots \wedge e_{j_i} \in L_i(\underline{x})$. Then we have the following sequence:

$$\begin{aligned}
&\partial_{i-1}^{L(\underline{x})}(\partial_i^{L(\underline{x})}(e_{j_1} \wedge \cdots \wedge e_{j_i})) \\
&= \partial_{i-1}^{L(\underline{x})} \left(\sum_{s=1}^{i} (-1)^{s+1} x_{j_s} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i} \right) \\
&= \sum_{s=1}^{i} (-1)^{s+1} x_{j_s} \partial_{i-1}^{L(\underline{x})} (e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i}) \\
&= \sum_{s=1}^{i} (-1)^{s+1} x_{j_s} \left(\sum_{r=1}^{s-1} (-1)^{r+1} x_{j_r} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_r} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i} \right. \\
&\qquad \left. + \sum_{r=s+1}^{i} (-1)^{r} x_{j_r} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge \widehat{e}_{j_r} \wedge \cdots \wedge e_{j_i} \right) \\
&= \sum_{s=1}^{i} \left(\sum_{r=1}^{s-1} (-1)^{s+r+2} x_{j_s} x_{j_r} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_r} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge e_{j_i} \right. \\
&\quad \left. + \sum_{r=s+1}^{i} (-1)^{r+s+1} x_{j_s} x_{j_r} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_s} \wedge \cdots \wedge \widehat{e}_{j_r} \wedge \cdots \wedge e_{j_i} \right) \\
&= 0.
\end{aligned}$$

The sum is split in the third step to deal with the two cases in which e_{j_r} may be on the right or the left of the e_{j_s} already removed. Note that each basis vector occurs in each list only once and that each list has an opposite sign. Because of this, the elements cancel each other out.

Therefore $L(\underline{x})$ is an R-complex. □

3. Day 3

We now turn our attention to proving the fact $L(\underline{x}) \cong K(\underline{x})$. We will aslo show $L(\underline{x}) \cong L(x_1) \otimes_R L(x_2) \otimes_R \cdots \otimes_R L(x_n)$. First we need a few definitions.

Definition 3.1. Define the submodule $\tilde{L}_i(\underline{x}) \subset L_i(\underline{x})$ to be:

$$\tilde{L}_i(\underline{x}) = (\{e_{j_1} \wedge \cdots \wedge e_{j_i} |\text{ excluding wedges with } e_n\}).$$

Note that with this definition we have $\tilde{L}_i(\underline{x}) \cong R^{\binom{n-1}{i}}$.

In addition, we can write $\tilde{L}(\underline{x})$ as the following sequence:

$$0 \to \begin{matrix} \tilde{L}_n(\underline{x}) \\ \oplus \\ \tilde{L}_{n-1}(\underline{x}) \end{matrix} \xrightarrow{\partial_n^{\tilde{L}(\underline{x})}} \begin{matrix} \tilde{L}_{n-1}(\underline{x}) \\ \oplus \\ \tilde{L}_{n-2}(\underline{x}) \end{matrix} \xrightarrow{\partial_{n-1}^{\tilde{L}(\underline{x})}} \cdots \xrightarrow{\partial_2^{\tilde{L}}} \begin{matrix} \tilde{L}_1(\underline{x}) \\ \oplus \\ \tilde{L}_0(\underline{x}) \end{matrix} \xrightarrow{\partial_1^{\tilde{L}(\underline{x})}} \begin{matrix} \tilde{L}_0(\underline{x}) \\ \oplus \\ 0 \end{matrix} \to 0.$$

The maps from the above sequence are defined as follows:

$$\partial_i^{\tilde{L}(\underline{x})} = \begin{pmatrix} \partial_i^{L(\underline{x})} & (-1)^{i+1} x_n \\ 0 & \partial_{i-1}^{L(\underline{x})} \end{pmatrix}.$$

Remark 3.2. $\tilde{L}(\underline{x})$ is an R-complex. This can be seen by the following computation:

$$\begin{aligned}
&\begin{pmatrix} \partial_{i-1}^{L(\underline{x})} & (-1)^{i+1}x_n \\ 0 & \partial_{i-2}^{L(\underline{x})} \end{pmatrix} \begin{pmatrix} \partial_{i}^{L(\underline{x})} & (-1)^{i+1}x_n \\ 0 & \partial_{i-1}^{L(\underline{x})} \end{pmatrix} \\
&\qquad = \begin{pmatrix} \partial_{i-1}^{L(\underline{x})}\partial_{i}^{L(\underline{x})} & (-1)^{i+1}x_n\partial_{i-1}^{L(\underline{x})} + (-1)^{i}x_n\partial_{i-1}^{L(\underline{x})} \\ 0 & \partial_{i-2}^{L(\underline{x})}\partial_{i-1}^{L(\underline{x})} \end{pmatrix} \\
&\qquad = 0.
\end{aligned}$$

Definition 3.3. Let

$$f_i : L_i(\underline{x}) \to \begin{matrix} \tilde{L}_i(\underline{x}) \\ \oplus \\ \tilde{L}_{i-1}(\underline{x}) \end{matrix}$$

be defined by

$$\underline{e} \mapsto \begin{cases} \begin{pmatrix} \underline{e} \\ 0 \end{pmatrix} & \text{if } \underline{e} \text{ does not contain } e_n \\ \begin{pmatrix} 0 \\ \underline{e} \wedge \widehat{e}_n \end{pmatrix} & \text{if } \underline{e} \text{ contains } e_n \end{cases}$$

where $\underline{e} = e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i}$ and $\widehat{e}_n$ indicates that e_n has been removed from $\underline{e}$.

Example 3.4. Consider the case where $n = 3$.

$$\tilde{L}(x_1, x_2, x_3) = 0 \to \begin{matrix} \tilde{L}(x_3) \\ \oplus \\ \tilde{L}(x_2) \end{matrix} \xrightarrow{\partial_3^{\tilde{L}(\underline{x})}} \begin{matrix} \tilde{L}(x_2) \\ \oplus \\ \tilde{L}(x_1) \end{matrix} \xrightarrow{\partial_2^{\tilde{L}(\underline{x})}} \begin{matrix} \tilde{L}(x_1) \\ \oplus \\ \tilde{L}(x_0) \end{matrix} \xrightarrow{\partial_1^{\tilde{L}(\underline{x})}} \begin{matrix} \tilde{L}(x_0) \\ \oplus \\ 0 \end{matrix} \to 0.$$

The basis vectors are as follows:

$\begin{matrix} \tilde{L}(x_3) \\ \oplus \\ \tilde{L}(x_2) \end{matrix}$ is generated by the vector $\begin{pmatrix} 0 \\ e_1 \wedge e_2 \end{pmatrix}$

$\begin{matrix} \tilde{L}(x_2) \\ \oplus \\ \tilde{L}(x_1) \end{matrix}$ is generated by the vectors $\begin{pmatrix} 0 \\ e_1 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2 \end{pmatrix}, \begin{pmatrix} e_1 \wedge e_2 \\ 0 \end{pmatrix}$

$\begin{matrix} \tilde{L}(x_1) \\ \oplus \\ \tilde{L}(x_0) \end{matrix}$ is generated by the vectors $\begin{pmatrix} e_1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$\begin{matrix} \tilde{L}(x_0) \\ \oplus \\ 0 \end{matrix}$ with basis $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Note also that $\tilde{L}(x_3) = 0$. From this we build the following commutative diagram:

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & L(x_3) & \xrightarrow{\partial_3^{L(\underline{x})}} & L(x_2) & \xrightarrow{\partial_2^{L(\underline{x})}} & L(x_1) & \xrightarrow{\partial_1^{L(\underline{x})}} & L(x_0) & \longrightarrow & 0 \\
 & & \downarrow f_3 & & \downarrow f_2 & & \downarrow f_1 & & \downarrow f_0 & & \\
0 & \longrightarrow & \begin{matrix}\tilde{L}(x_3)\\ \oplus \\ \tilde{L}(x_2)\end{matrix} & \xrightarrow{\partial_3^{\tilde{L}(\underline{x})}} & \begin{matrix}\tilde{L}(x_2)\\ \oplus \\ \tilde{L}(x_1)\end{matrix} & \xrightarrow{\partial_2^{\tilde{L}(\underline{x})}} & \begin{matrix}\tilde{L}(x_1)\\ \oplus \\ \tilde{L}(x_0)\end{matrix} & \xrightarrow{\partial_1^{\tilde{L}(\underline{x})}} & \begin{matrix}\tilde{L}(x_0)\\ \oplus \\ 0\end{matrix} & \longrightarrow & 0.
\end{array}$$

We need to show that the diagram commutes and that the maps f_3, f_2, and f_1 are isomorphisms. To show that the diagram commutes we need only work with the basis vectors. To show that the first square commutes, we compute $\partial_3^{\tilde{L}(\underline{x})} \circ f_3$ and $f_2 \circ \partial_3^{L(\underline{x})}$ as follows:

$$\begin{aligned}
\partial_3^{\tilde{L}(\underline{x})}(f_3(e_1 \wedge e_2 \wedge e_3)) &= \begin{pmatrix} \partial_3^{L(\underline{x})} & (-1)^{1+3}x_3 \\ 0 & \partial_2^{L(\underline{x})} \end{pmatrix} \begin{pmatrix} 0 \\ e_1 \wedge e_2 \end{pmatrix} \\
&= \begin{pmatrix} x_3 e_1 \wedge e_2 \\ \partial_2^{L(\underline{x})}(e_1 \wedge e_2) \end{pmatrix} \\
&= \begin{pmatrix} x_3 e_1 \wedge e_2 \\ x_1 e_2 - x_2 e_1 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
f_2(\partial_3^{L(\underline{x})}(e_1 \wedge e_2 \wedge e_3)) &= f_2(x_1 e_2 \wedge e_3 - x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2) \\
&= f_2(x_1 e_2 \wedge e_3) - f_2(x_2 e_1 \wedge e_3) + f_2(x_3 e_1 \wedge e_2) \\
&= \begin{pmatrix} 0 \\ x_1 e_2 \end{pmatrix} - \begin{pmatrix} 0 \\ x_2 e_1 \end{pmatrix} + \begin{pmatrix} x_3 e_1 \wedge e_2 \\ 0 \end{pmatrix}.
\end{aligned}$$

For the next square we have the following with respect to the basis element $e_2 \wedge e_3$:

$$\begin{aligned}
\partial_2^{\tilde{L}(\underline{x})}(f_2(e_2 \wedge e_3)) &= \begin{pmatrix} \partial_2^{L(\underline{x})} & (-1)^{1+2}x_3 \\ 0 & \partial_1^{L(\underline{x})} \end{pmatrix} \begin{pmatrix} 0 \\ e_2 \end{pmatrix} \\
&= \begin{pmatrix} -x_3 e_2 \\ x_2 \end{pmatrix} \\
f_1(\partial_2^{L(\underline{x})}(e_2 \wedge e_3)) &= f_1(x_2 e_3 - x_3 e_2) \\
&= \begin{pmatrix} 0 \\ x_2 \end{pmatrix} + \begin{pmatrix} -x_3 e_2 \\ 0 \end{pmatrix}.
\end{aligned}$$

It is left to the reader to check the remaining basis elements and squares. It is important to note that the sign change is necessary.

Proposition 3.5. *We have* $L(\underline{x}) \cong \tilde{L}(\underline{x})$.

Proof. Since the maps are already defined, it remains to show that the following diagram commutes:

$$\begin{array}{ccc} L_i(\underline{x}) & \xrightarrow{\partial_i^{L(\underline{x})}} & L_{i-1}(\underline{x}) \\ \cong \downarrow f_i & & \cong \downarrow f_{i-1} \\ \tilde{L}_i(\underline{x}) & & \tilde{L}_{i-1}(\underline{x}) \\ \oplus & \xrightarrow{\partial_i^{\tilde{L}(\underline{x})}} & \oplus \\ \tilde{L}_{i-1}(\underline{x}) & & \tilde{L}_{i-2}(\underline{x}). \end{array}$$

We proceed by cases.

Case 1: $\underline{e}$ contains e_n.

$$\begin{aligned} \partial_i^{\tilde{L}(\underline{x})}(f_i(\underline{e})) &= \partial_i^{\tilde{L}(\underline{x})}\left(\begin{pmatrix} 0 \\ \underline{e} \wedge \widehat{e}_n \end{pmatrix}\right) \\ &= \begin{pmatrix} (-1)^{i+1} x_n \underline{e} \wedge \widehat{e}_n \\ \sum_{l=1}^{i-1} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \widehat{e}_n \wedge \cdots \wedge e_{j_i} \end{pmatrix} \\ f_{i-1}(\partial_i^{L(\underline{x})}(\underline{e})) &= f_{i-1}\left(\sum_{l=1}^{i-1} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \cdots \wedge e_{j_i}\right) \\ &= f_{i-1}\Bigg((-1)^{i+1} x_n \underline{e} \wedge \widehat{e}_n \\ &\qquad + \sum_{l=1}^{i-1} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \cdots \wedge e_{j_i}\Bigg) \\ &= \begin{pmatrix} (-1)^{i+1} x_n \underline{e} \wedge \widehat{e}_n \\ 0 \end{pmatrix} \\ &\qquad + \begin{pmatrix} 0 \\ \sum_{l=1}^{i-1} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \widehat{e}_n \wedge \cdots \wedge e_{j_i} \end{pmatrix} \end{aligned}$$

Case 2: $\underline{e}$ does not contain e_n.

$$\begin{aligned} \partial_i^{\tilde{L}(\underline{x})}(f_i(\underline{e})) &= \begin{pmatrix} \partial_i^{L(\underline{x})} & (-1)^{i+1} x_n \\ 0 & \partial_{i-1}^{L(\underline{x})} \end{pmatrix} \begin{pmatrix} \underline{e} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \sum_{l=1}^{i} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \cdots \wedge e_{j_i} \\ 0 \end{pmatrix} \\ f_{i-1}(\partial_i^{L(\underline{x})}(\underline{e})) &= f_{i-1}\left(\sum_{l=1}^{i} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \cdots \wedge e_{j_i}\right) \\ &= \begin{pmatrix} \sum_{l=1}^{i} (-1)^{l+1} x_{j_l} e_{j_1} \wedge \cdots \wedge \widehat{e}_{j_l} \wedge \cdots \wedge e_{j_i} \\ 0 \end{pmatrix} \end{aligned}$$

Hence we conclude that the diagram commutes. □

Remark 3.6. Note that we also have the map:

$$L_i(\underline{x}) \cong R^{\binom{n}{i}} \xrightarrow{f_i} \begin{matrix} R^{\binom{n-1}{i}} \\ \oplus \\ R^{\binom{n-1}{i-1}} \end{matrix} \cong \begin{matrix} \tilde{L}_i(\underline{x}) \\ \oplus \\ \tilde{L}_{i-1}(\underline{x}) \end{matrix}.$$

The definition we have for the exterior algebra can be written as a tensor product.

Proposition 3.7. *There exists an isomorphism*

$$L(\underline{x}) \cong L(x_1) \otimes \cdots \otimes L(x_n).$$

Proof. Base case: $\underline{x} = x_1$. We have $L(x_1) \cong L(x_1)$.

Inductive step: Assume that $L(\underline{x}') \cong L(x_1) \otimes \cdots \otimes L(x_{n-1})$. Note $L(x_n) \cong K(x_n)$. In Proposition 1.9 we proved that $K(y) \otimes_R X = \operatorname{Cone}(\mu_X^y)$ with differentials given by multiplication by the matrix:

$$\begin{pmatrix} \partial_i^X & y \\ 0 & -\partial_{i-1}^X \end{pmatrix}.$$

For this proof we use $y = x_n$. We show

$$\tilde{L}(\underline{x}) \cong L(x_1) \otimes_R L(x_2) \otimes_R \cdots \otimes_R L(x_n) \cong L(\underline{x}') \otimes_R L(x_n).$$

Define the map $g_i : \tilde{L}(\underline{x}) \to L(\underline{x}') \otimes L(x_n)$ by multiplication by the matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & (-1)^{i+1} \end{pmatrix}.$$

Then we have the commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\begin{matrix}\tilde{L}_n(\underline{x})\\ \oplus \\ \tilde{L}_{n-1}(\underline{x})\end{matrix} & \longrightarrow & \begin{matrix}\tilde{L}_{n-1}(\underline{x})\\ \oplus \\ \tilde{L}_{n-2}(\underline{x})\end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix}\tilde{L}_1(\underline{x})\\ \oplus \\ \tilde{L}_0(\underline{x})\end{matrix} & \longrightarrow & \begin{matrix}\tilde{L}_0(\underline{x})\\ \oplus \\ 0\end{matrix} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\begin{matrix}0\\ \oplus \\ L_{n-1}(\underline{x}')\end{matrix} & \longrightarrow & \begin{matrix}L_{n-1}(\underline{x}')\\ \oplus \\ L_{n-2}(\underline{x}')\end{matrix} & \longrightarrow \cdots \longrightarrow & \begin{matrix}L_1(\underline{x}')\\ \oplus \\ L_0(\underline{x}')\end{matrix} & \longrightarrow & \begin{matrix}L_0(\underline{x}')\\ \oplus \\ 0\end{matrix}
\end{array}$$

Each component of $L(\underline{x}') \otimes L(x_n)$ has $L_i(\underline{x}') \cong R^{\binom{n-1}{i}}$. The maps down are one to one and onto, therefore they are isomorphisms. Observe that $\partial_i^{\tilde{L}(\underline{x})} = \partial_i^{L(\underline{x}')}$. Showing that the squares commute is left as an exercise for the reader. Applying the inductive hypothesis we have

$$L(\underline{x}) \cong \tilde{L}(\underline{x}) \cong L(\underline{x}') \otimes L(x_n) \cong L(x_1) \otimes \cdots \otimes L(x_n).$$

□

The next corollary proves that Definitions 1.3 and 2.2 of the Koszul complex are equivalent.

Corollary 3.8. *We have* $L(\underline{x}) \cong K(\underline{x})$.

Proof. For any n we have $K(x_n) = L(x_n)$. This explains the second step in the following sequence

$$L(\underline{x}) \cong L(x_1) \otimes \cdots \otimes L(x_n) \cong K(x_1) \otimes \cdots \otimes K(x_n) \cong K(\underline{x}).$$

The first step is from Proposition 3.7 and the third step is from Proposition 1.9. Thus $L(\underline{x}) \cong K(\underline{x})$. □

4. Day 4

Now we show that the Koszul complex is independent of the generating sequence. In other words, if $(R, \mathfrak{m}, k)$ is a local ring and $a_1, \cdots, a_n, b_1, \cdots, b_n \in R$ with $(\underline{a})R = (\underline{b})R$, then $K(\underline{a}) = K(\underline{b})$.

It should be noted that for the proof given here it is necessary that the ring is local. Although the Koszul complex is probably independent of the generating sequence in the general case, we do not have at this moment a proof concerning the general case. For the proof we present here, it is crucial that the generating sequences we work with be of the same length. If the sequences are minimal generating sequences, their length must be equal. The proof given does not require that the generating sequences be minimal, only that they have equal length.

Remark 4.1. Recall that $K_j(\underline{a}) \cong R^{\binom{n}{j}}$ with basis vectors $\{e_{i_1} \wedge \cdots \wedge e_{i_j} | 1 \leqslant i_1 < \cdots < i_j \leqslant n\}$ and we write $e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_r} \wedge \cdots \wedge e_{i_j}$ to indicate that e_{i_r} has been omitted from the wedge product.

Definition 4.2. Let $\underline{a} = a_1, \cdots, a_n \in R$ be a sequence. Fix $e_{i_1} \wedge \cdots \wedge e_{i_j} \in K_{j-1}(\underline{a})$ such that $1 \leqslant i_2 < \cdots < i_j \leqslant n$. Let $1 \leqslant l \leqslant n$ and define $e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \in K_j(\underline{a})$ to be as follows:

$$e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} = \begin{cases} 0 & \text{if } l \in \{i_2, \cdots, i_j\} \\ e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} & \text{if } l < i_2 \\ -e_{i_2} \wedge e_l \wedge e_{i_3} \wedge \cdots \wedge e_{i_j} & \text{if } i_2 < l < i_3 \\ \vdots & \\ (-1)^{s+1} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_j} & \text{if } i_s < l < i_{s+1} \\ \vdots & \\ (-1)^{j+1} e_{i_2} \wedge \cdots \wedge e_{i_j} \wedge e_l & \text{if } i_j < l. \end{cases}$$

Each time we permute the order of the wedge product, we multiply by -1. Thus an even number of permutations yields a plus sign and an odd number of permutations yields a negative sign.

Lemma 4.3. *Let $\underline{a} = a_1, \cdots, a_n \in R$ be a sequence. Then*

$$\partial_j^{K(\underline{a})}(e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = x_l e_{i_2} \wedge \cdots \wedge e_{i_j} + \sum_{t=2}^{j} (-1)^{t+1} x_{i_t} e_l \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j}.$$

Proof. Case 1: $l \in \{i_2, \cdots, i_j\}$. We see by Definiton 4.2 that

$$\partial_j^{K(\underline{a})}(e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = 0.$$

Assume that $l = i_s$ for some s. For the right hand side of the desired equality we have the following:

$$\begin{aligned}
\text{RHS} &= x_{i_s} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_j} \\
&\qquad + (-1)^{s+1} x_{i_s} e_{i_s} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_s} \wedge \cdots \wedge e_{i_j} \\
&\qquad + \sum_{t \geqslant 2, t \neq s} (-1)^{t+1} x_{i_t} e_{i_s} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j} \\
&= x_{i_s} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_j} \\
&\qquad + (-1)^{s+1} x_{i_s} e_{i_s} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_s} \wedge \cdots \wedge e_{i_j} \\
&= x_{i_s} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_j} \\
&\qquad + (-1)^{s+1} (-1)^s x_{i_s} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_j} \\
&= 0.
\end{aligned}$$

In the first step we separate the sum where $t = s$ from the larger sum. In step two, the sum $\sum_{t \geqslant 2, t \neq s} (-1)^{t+1} x_{i_t} e_{i_s} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j}$ vanishes since $e_s \wedge e_s = 0$ appears in every term of the sum. By permuting the order of the wedge product in step three, we have a sign change. This produces the cancellation in step four.

Case 2: $i_s < l < i_{s+1}$ for some $s \in \{2, \ldots, j\}$. For the left hand side of the desired equality we have the following:

$$\begin{aligned}
\text{LHS} &= (-1)^{s+1} \partial_j^{K(\underline{a})} (e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_j}) \\
&= (-1)^{s+1} \left(\sum_{t=2}^{s} (-1)^t x_{i_t} e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_j} \right. \\
&\quad + (-1)^{s+1} x_l e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_j} \\
&\quad \left. + \sum_{t=s+1}^{j} (-1)^{t+1} x_{i_t} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j} \right).
\end{aligned}$$

The second step splits up the sum into the cases where e_l is the the right, between, and to the left of the wedge $e_s \wedge e_{s+1}$. For the right hand side of the desired equality we have the following:

$$\begin{aligned}
\text{RHS} &= x_l e_{i_2} \wedge \cdots \wedge e_{i_j} \\
&\qquad + \sum_{t=2}^{s} (-1)^{t+1} x_{i_t} e_l \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_s} \wedge \cdots \wedge e_{i_j} \\
&\qquad + \sum_{t=s+1}^{j} (-1)^{t+1} x_{i_t} e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_{i_{s+1}} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j} \\
&= x_l e_{i_2} \wedge \cdots \wedge e_{i_j} \\
&\qquad + \sum_{t=2}^{s} (-1)^{t+1} (-1)^s x_{i_t} e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge e_{i_j} \\
&\qquad + \sum_{t=s+1}^{j} (-1)^{t+1} (-1)^{s+1} x_{i_t} e_{i_2} \wedge \cdots \wedge e_{i_s} \wedge e_l \wedge e_{i_{s+1}} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j}.
\end{aligned}$$

For the second step, the element e_l is moved in two of the sums and the wedge product is multiplied by the necessary sign change. By matching up the terms from the LHS and RHS, we see that the two are equal.

Cases 3 and 4: $l < i_2$ or $i_j < l$. These cases are similar to Case 2 and are left to the reader as an exercise. ☐

Lemma 4.4. *Let $\underline{b} = b_1, \cdots, b_n \in R$ and set $a_1 = \sum_{l=1}^{n} r_l b_l$ such that $r_l \in R$ for all $l = 1, \cdots, n$. Let $\underline{b}^\dagger = a_1, b_2, \cdots, b_n$. Let the basis vectors of $K(\underline{b}^\dagger)$ be in terms of the elements e_i, and let the basis vectors of $K(\underline{b})$ be in terms of the elements f_i. Define the map*

$$\psi : K(\underline{b}^\dagger) \to K(\underline{b})$$

as follows:

$$\begin{array}{rl} \textit{in degree 0} & 1 \longmapsto 1 \\ \textit{in degree 1} & \begin{cases} e_1 \longmapsto \sum_{l=1}^{n} r_l f_l & \\ e_i \longmapsto f_i & \textit{for } i \geqslant 2 \end{cases} \\ \textit{in degree } j \geqslant 2 & \begin{cases} e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \longmapsto \sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge f_{i_j} \\ e_{i_1} \wedge \cdots \wedge e_{i_j} \longmapsto f_{i_1} \wedge \cdots \wedge f_{i_j}. \end{cases} \end{array}$$

Then ψ is a chain map.

Proof. To show that ψ is a chain map, it suffices to show that the diagrams commute with respect to the basis elements.

$$\begin{array}{ccccccc} \cdots & \longrightarrow & K_j(\underline{b}^\dagger) & \xrightarrow{\partial_j^{K(\underline{b}^\dagger)}} & K_{j-1}(\underline{b}^\dagger) & \longrightarrow & \cdots \\ & & \downarrow{\scriptstyle \psi_j} & & \downarrow{\scriptstyle \psi_{j-1}} & & \\ \cdots & \longrightarrow & K_j(\underline{b}) & \xrightarrow{\partial_j^{K(\underline{b})}} & K_{j-1}(\underline{b}) & \longrightarrow & \cdots \end{array}$$

For the case where $j = 1$, we have the following diagram:

$$\begin{array}{ccc} R^n & \xrightarrow{\partial_1^{K(\underline{b}^\dagger)}} & R \\ \downarrow{\scriptstyle \psi_1} & & \downarrow{\scriptstyle id} \\ R^n & \xrightarrow{\partial_1^{K(\underline{b})}} & R. \end{array}$$

Below are the basis vectors. On the right is the case where $i = 1$ and on the left is the case where $i \geqslant 2$

$$\begin{array}{ccc} e_1 & \longmapsto & a_1 = \sum_{l=1}^{n} r_l b_l \\ \downarrow & & \downarrow \\ \sum_{l=1}^{n} r_l f_l & \longmapsto & \sum_{l=1}^{n} r_l b_l \end{array} \qquad \begin{array}{ccc} e_i & \longmapsto & b_i \\ \downarrow & & \downarrow \\ f_i & \longmapsto & b_i. \end{array}$$

When $j \geqslant 2$, we have two cases. We show first the case where $i_1 > 1$:

$$\begin{array}{ccc} e_{i_1} \wedge \cdots \wedge e_{i_j} & \longmapsto & \sum_{t=1}^{j}(-1)^{t+1} b_{i_t} e_{i_1} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j} \\ \downarrow & & \downarrow \\ f_{i_1} \wedge \cdots \wedge f_{i_j} & \longmapsto & \sum_{t=1}^{j}(-1)^{t+1} b_{i_t} f_{i_1} \wedge \cdots \wedge \widehat{f}_{i_t} \wedge \cdots \wedge f_{i_j}. \end{array}$$

Next we show the case where $i_1 = 1$.

$$\begin{aligned} \psi_{j-1}(\partial_j^{K(\underline{b}^\dagger)}(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j})) &= \psi_{j-1}\Bigg(a_1 e_{i_2} \wedge \cdots \wedge e_{i_j} \\ &\quad + \sum_{t=2}^{j}(-1)^{t+1} b_{i_t} e_1 \wedge e_{i_2} \wedge \cdots \wedge \widehat{e}_{i_t} \wedge \cdots \wedge e_{i_j}\Bigg) \\ &= a_1 f_{i_2} \wedge \cdots \wedge f_{i_j} + \\ &\quad \sum_{t=2}^{j}(-1)^{t+1} b_{i_t}\left(\sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge \widehat{f}_{i_t} \wedge \cdots \wedge f_{i_j}\right) \end{aligned}$$

$$\begin{aligned} \partial_j^{K(\underline{b})}(\psi_j(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j})) &= \partial_j^{K(\underline{b})}\left(\sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge f_{i_j}\right) \\ &= \sum_{l=1}^{n} r_l \left(\partial_j^{K(\underline{b})}(f_l \wedge f_{i_2} \wedge \cdots \wedge f_{i_j})\right) \\ &= \sum_{l=1}^{n} r_l \Bigg(b_l f_{i_2} \wedge \cdots \wedge f_{i_j} \\ &\quad + \sum_{t=2}^{j}(-1)^{t+1} b_{i_t} f_l \wedge f_{i_2} \wedge \cdots \wedge \widehat{f}_{i_t} \wedge \cdots \wedge f_{i_j}\Bigg) \\ &= \sum_{l=1}^{n} r_l b_l f_{i_2} \wedge \cdots \wedge f_{i_j} + \\ &\quad \sum_{t=2}^{j}(-1)^{t+1} b_{i_t}\left(\sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge \widehat{f}_{i_t} \wedge \cdots \wedge f_{i_j}\right) \\ &= a_1 f_{i_2} \wedge \cdots \wedge f_{i_j} + \\ &\quad \sum_{t=2}^{j}(-1)^{t+1} b_{i_t}\left(\sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge \widehat{f}_{i_t} \wedge \cdots \wedge f_{i_j}\right) \end{aligned}$$

where the third step in the second sequence above follows from Lemma 4.3. □

Lemma 4.5. *Let $b_1, \cdots, b_n, r_1, \cdots, r_n \in R$ and assume that r_1 is a unit. Define $a_1 = \sum_{l=1}^{n} r_l b_l$ and $\underline{b}^\dagger = a_1, b_2, \cdots, b_n$. Then the map*

$$\psi : K(\underline{b}^\dagger) \to K(\underline{b})$$

is an isomorphism.

Proof. We proceed by constructing the inverse to ψ. Note that ψ is well-defined. Since r_1 is a unit, we can solve the equation $a_1 = \sum_{l=1}^{n} r_l b_l$ for b_1, obtaining

$b_1 = r_1^{-1}(a_1 - \sum_{l=1}^{n} r_l b_l)$. Using a procedure similar to that in Lemma 4.4, the sequence $a_1, b_2, \cdots, b_n$ gives rise to the sequence $b_1, b_2, \cdots, b_n$. There exists a chain map

$$\Phi : K(\underline{b}) \to K(\underline{b}^\dagger)$$

that maps the basis vectors as follows:

$$\begin{aligned} f_1 &\longmapsto r_1^{-1} e_1 - \sum_{l=2}^{n} r_1^{-1} r_l e_l \\ f_i &\longmapsto e_i \quad \text{for} \quad i \geqslant 2. \end{aligned}$$

The wedges are defined with a procedure similar to that in Lemma 4.4.

Next we need to show that the compositions of the maps Φ and ψ yield the appropriate identity. We show the case for $\Phi \circ \psi$ and leave the other case as an exercise for the reader. It suffices to check the basis vectors with the following computations:

$$\begin{aligned} \Phi(\psi(e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j})) &= \Phi\left(\sum_{l=1}^{n} r_l f_l \wedge f_{i_2} \wedge \cdots \wedge f_{i_j}\right) \\ &= r_1 \Phi(f_1 \wedge f_{i_2} \wedge \cdots \wedge f_{i_j}) + \sum_{l=2}^{n} r_l \Phi(f_l \wedge f_{i_2} \wedge \cdots \wedge f_{i_j}) \\ &= r_1 \left(r_1^{-1} e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} - \sum_{l=2}^{n} r_1^{-1} r_l e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}\right) \\ &\quad + \sum_{l=2}^{n} r_l e_l \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \\ &= e_1 \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \\ \Phi(\psi(e_1)) &= \Phi\left(\sum_{l=1}^{n} r_l f_l\right) \\ &= r_1 \Phi(f_1) + \Phi\left(\sum_{l=2}^{n} r_l f_l\right) \\ &= r_1 \left(r_1^{-1} e_1 - \sum_{l=2}^{n} r_1^{-1} r_l e_l\right) + \sum_{l=2}^{n} r_l e_l \\ &= e_1. \end{aligned}$$

□

Remark 4.6. For following theorem, we use Nakayama's lemma; this popular lemma can be found in many sources including [1, Proposition 2.6], [5, Theorem 2.2], or [6, Proposition 4.51]. For this purpose we assume that the ring $(R, \mathfrak{m}, k)$ is local. Since there exists a version of Nakayama's lemma for the non-local case, it would also suffice to assume that the sequences used are contained in the Jacobson radical. As another alternative, there exists a version of Nakayama's lemma for graded rings; see [2, Exercise 1.5.24].

Theorem 4.7. *Assume that $(R, \mathfrak{m}, k)$ is local. Let $\underline{a} = a_1, \cdots, a_n \in R$ and $\underline{b} = b_1, \cdots, b_n \in R$ such that $I = (\underline{a})R = (\underline{b})R$ is a minimal generating sequence. Then $K(\underline{a}) \cong K(\underline{b})$.*

Proof. Step 1: We claim that for all $i = 1, \cdots, n$ we have $a_i \notin (a_1, \cdots, a_{i-1})R + \mathfrak{m}I$. Note that this is a version of Nakayama's lemma. Since $\overline{a}_1, \cdots \overline{a}_n \in I/\mathfrak{m}I$ is a basis, we conclude that $\overline{a}_1, \cdots, \overline{a}_{i-1}, \overline{a}_i$ are linearly independent.

Step 2: We claim that there exists $r_{11}, \cdots, r_{1n} \in R$ such that

$$a_1 = \sum_{j=1}^{n} r_{ij} b_j \tag{4.7.1}$$

where r_{ij} a unit for some j. If not, then $r_{ij} \in \mathfrak{m}$ for all j. This implies that $a_1 \in \mathfrak{m}I$. But since a_1 is a minimal generator for I, this is a contradiction of Nakayama's lemma.

Step 3: Reorder $b_1, \cdots, b_n$ such that r_{11} is a unit. We claim $I = (a_1, b_2, \cdots, b_n)R$.

$\supseteq$: This containment is evident.

$\subseteq$: Since $I = (\underline{b})R$, it suffices to show that $b_j \in (a_1, b_2, \cdots, b_n)R$ for all $j = 1, \cdots, n$. When $j \geqslant 2$, this is clear. In the case $j = 1$, since r_{11} is a unit we can solve equation (4.7.1) for b_1. Then $b_1 = r_{11}^{-1}(a_1 - \sum_{j=2}^{n} r_{1j} b_j) \in (a_1, b_2, \cdots, b_n)R$.

Step 4: Since $a_2 \in I = (a_1, b_2, \cdots, b_n)R$, we have the equation $a_2 = r_{21}a_1 + \sum_{j=2}^{n} r_{2j} b_j$. We claim that there exists a $j \geqslant 2$ such that r_{2j} is a unit. If not, then $a_2 \in (a_1)R + \mathfrak{m}I$. This contradicts Step 1 when $i = 2$.

Step 5: Reorder $b_2, \cdots, b_n$ such that r_{22} is a unit. Using the method in Step 3, we conclude that $I = (a_1, a_2, b_3, \cdots, b_n)R$.

Step 6: In this step we use what is informally known as Roger Induction (see Sather-Wagstaff for an explanation). We reorder the b_j's so that $a_i = \sum_{j=1}^{i-1} r_{ij} a_j + \sum_{j=i}^{n} r_{ij} b_j$ where r_{ij} is a unit and $I = (a_1, \cdots, a_{i-1}, b_i, \cdots, b_n)R$.

Step 7: Lemma 4.5 explains the even steps in following sequence:

$$\begin{aligned} K(\underline{b}) &= K(b_1, \cdots, b_n) \\ &\cong K(a_1, b_2, \cdots, b_n) \\ &\cong K(b_2, a_1, b_3, \cdots, b_n) \\ &\cong K(a_2, a_1, b_3, \cdots, b_n) \\ &\quad \vdots \\ &\cong K(a_1, a_2, \cdots, a_n) \\ &= K(\underline{a}). \end{aligned}$$

The first and last steps are by definition. The remaining steps follow from Definition 1.8. □

When we remove the assumption that the generating sequences are minimal from the above theorem, the result is the following.

Theorem 4.8. *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\underline{a} = a_1, \cdots, a_n \in R$ and $\underline{b} = b_1, \cdots, b_n \in R$ such that $I = (\underline{a})R = (\underline{b})R$ is a generating sequence. Then $K(\underline{a}) \cong K(\underline{b})$.*

Proof. If the sequences are minimal generating sequences, then we are done. Assume then that the sequences are not minimal. Nakayama's lemma implies that we can reorder any sequence so that the initial elements are minimal and the remaining elements are redundant. Reorder $\underline{a}$ and $\underline{b}$ so that $a_1, \cdots, a_m$ and $b_1, \cdots, b_m$ are minimal generating sequences.

We show that $K(\underline{b}) \cong K(b_1, \cdots, b_m, 0, \cdots, 0)$ where there are $n-m$ zeros. Since $b_{m+1} \in I = (b_1, \cdots, b_m)R$, we can write $b_{m+1} = \sum_{j=1}^{m} r_j b_j$ for some $r_1, \ldots, r_m \in R$. Then $0 = \sum_{j=1}^{m} r_j b_j + (-1)b_{m+1}$. Since -1 is a unit, Lemma 4.5 yields the following sequence:

$$\begin{aligned} K(b_1, \cdots, b_m, b_{m+1}, \cdots, b_n) &\cong K(b_1, \cdots, b_m, 0, b_{m+2}, \cdots, b_n) \\ &\vdots \\ &\cong K(b_1, \cdots, b_m, 0, \cdots, 0). \end{aligned}$$

This explains the first and fifth steps in the following sequence:

$$\begin{aligned} K(\underline{b}) &\cong K(b_1, \cdots, b_m, 0, \cdots, 0) \\ &\cong K(b_1, \cdots, b_m) \otimes_R K(0, \cdots, 0) \\ &\cong K(a_1, \cdots, a_m) \otimes_R K(0, \cdots, 0) \\ &\cong K(a_1, \cdots, a_m, 0, \cdots, 0) \\ &\cong K(\underline{a}). \end{aligned}$$

The second and fourth steps are by definition and the third step follows from Theorem 4.7. $\square$

The next example shows that although it is not necessary that the generating sequences $\underline{a}$ and $\underline{b}$ be minimal, it is necessary that the sequences be of the same length for $K(\underline{a})$ to be isomorphic to $K(\underline{b})$.

Example 4.9. Consider the complexes $K(0)$ and $K(0,0)$. These complexes are not isomorphic nor are they quasiisomorphic. We have the following:

$$K(0) = 0 \to 0 \to R \xrightarrow{0} R \to 0 \cong R \oplus \Sigma R$$

$$K(0,0) = 0 \to R \xrightarrow{0} R^2 \xrightarrow{0} R \to 0 \cong R \oplus \Sigma R^2 \oplus \Sigma R^2.$$

The homologies are different as well:

$$H_i(K(0)) \cong K_i(0) \cong R^{\binom{1}{i}}$$

$$H_i(K(0,0)) \cong K_i(0,0) \cong R^{\binom{2}{i}}.$$

Thus $K(0)$ and $K(0,0)$ are not even quasiisomorphic much less isomorphic.

Example 4.10. The Koszul complexes $K(x)$ and $K(x,0)$ are not isomorphic nor are they quasiisomorphic.

Lemma 4.11. *If $u \in R$ is a unit, then $K(u, a_2, \cdots, a_n)$ is exact.*

Proof. Using the mapping cone definition of the Koszul complex, we have

$$K(u, a_2, \cdots, a_n) \cong K(u) \otimes_R K(a_2, \cdots, a_n) \cong \mathrm{Cone}(\mu^u_{K(a_2,\cdots,a_n)}).$$

Since u is a unit, the map $\mu^u_{K(a_2,\cdots,a_n)}$ is an isomorphism. Thus $\mu^u_{K(a_2,\cdots,a_n)}$ is a quasiisomorphism and $\mathrm{Cone}(\mu^u_{K(a_2,\cdots,a_n)})$ is exact. $\square$

Theorem 4.12. *Let $a_1, \cdots, a_n, b_1, \cdots, b_n \in R$ such that $(\underline{a})R = (\underline{b})R$. Then we have the following:*

(a) *For all* $\mathfrak{p} \in \operatorname{Spec}(R)$*, we have* $H_i(K(\underline{a}))_{\mathfrak{p}} \cong H_i(K(\underline{b}))_{\mathfrak{p}}$ *for all* i*.*
(b) *Fix an integer* i*. Then* $H_i(K(\underline{a})) = 0$ *if and only if* $H_i(K(\underline{b})) = 0$*.*

Proof. For part (a), we have the following sequence:

$$H_i(K(\underline{a}))_{\mathfrak{p}} \cong H_i(K(\underline{a})_{\mathfrak{p}}) \cong H_i(K(\underline{b})_{\mathfrak{p}}) \cong H_i(K(\underline{b}))_{\mathfrak{p}}.$$

The first and last isomorphisms are $R_{\mathfrak{p}}$-module isomorphisms. For the second step, the equality $(\underline{a})R = (\underline{b})R$ implies that $(\underline{a})R_{\mathfrak{p}} = (\underline{b})R_{\mathfrak{p}}$. Theorem 4.8 implies that $K^{R_{\mathfrak{p}}}(\underline{a}) \cong K^{R_{\mathfrak{p}}}(\underline{b})$ yielding the desired isomorphism.

For part (b), we use the fact that homology is a local property. Part (a) implies that $H_i(K(\underline{a})) = 0$ if and only if $H_i(K(\underline{a}))_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if $H_i(K(\underline{b}))_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ if and only if $H_i(K(\underline{b})) = 0$. □

5. Day 5

Our next goal is to show that $K(\underline{x})$ is self-dual, that is, we show

$$\Sigma^n \operatorname{Hom}_R(K(\underline{x}), R) \cong K(\underline{x}).$$

This is accomplished in Theorem 7.1. Before we get to that, we introduce some background information about homomorphisms of complexes together with shifts of complexes, and some natural maps. Much of the following information can be found in [4].

Remark 5.1. Let X and Y be R-complexes. Then $\operatorname{Hom}_R(X, Y)$ is an R-complex where the lth module is defined as

$$\operatorname{Hom}_R(X, Y)_l = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l}).$$

The maps of the complex $\operatorname{Hom}_R(X, Y)$ act on a family of R-module homomorphisms $(\alpha_p) \in \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l})$ where $\alpha_p : X_p \to Y_{p+l}$. These maps are defined as follows

$$\partial_l^{\operatorname{Hom}_R(X,Y)} : \operatorname{Hom}_R(X, Y)_l \to \operatorname{Hom}_R(X, Y)_{l-1}$$

where $\partial_l^{\operatorname{Hom}_R(X,Y)}((\alpha_p)) = (\partial_{p+l}^Y \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X)$ and

$$(\partial_{p+l}^Y \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X) : X_p \to Y_{p+l-1}.$$

See [4, (3.1)] for details showing that the above map is a chain map.

Example 5.2. Compute $\Sigma^1 \operatorname{Hom}_R(K(x), R) = \Sigma \operatorname{Hom}_R(K(x), R)$. We have the following sequence

$$\begin{array}{lccccccc} K(x): & 0 \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow 0 \\ \text{degree}: & & 1 & & 2. & \end{array}$$

We compute the components of the complex

$$\operatorname{Hom}_R(K(x), R)_l = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(K(x)_p, R_{p+l}).$$

We have $R_{p+l} \neq 0$ when $p + l = 0$ and $K(x)_p \neq 0$ when $p = 0$ or $p = 1$. Thus there are two cases which we need to check. In the case where $p = 0$ and $l = 0$, we have the following

$$\operatorname{Hom}_R(K(x), R)_0 = \operatorname{Hom}_R(K(x)_0, R_0) = \operatorname{Hom}_R(R, R) \cong R.$$

For the case $p = 1$ and $l = -1$, we have the following

$$\operatorname{Hom}_R(K(x), R)_{-1} = \operatorname{Hom}_R(K(x)_1, R_0) = \operatorname{Hom}_R(R, R) \cong R.$$

For the maps, we have by definition

$$\partial_0^{\operatorname{Hom}_R(K(x),R)}((\alpha_p)) = (\partial_p^R \circ \alpha_p - (-1)^0 \alpha_{p-1} \circ \partial_p^{K(x)}).$$

Since the map $\partial_p^R = 0$ for all p and $\partial_p^{K(x)} = 0$ for all $p \neq 1$, we have the following

$$\partial_0^{\operatorname{Hom}_R(K(x),R)}((\alpha_p)) = (\cdots, 0, -\alpha_0 \circ \partial_1^{K(x)}, 0, \cdots).$$

Thus we have $-\alpha_0 \circ x_1 = -\partial_0^R$ and hence

$$\begin{array}{llcccc} \operatorname{Hom}_R(K(x), R): & 0 \longrightarrow & R & \xrightarrow{-x} & R & \longrightarrow 0 \\ \text{degree}: & & 0 & & 1. & \end{array}$$

Then we have

$$\begin{aligned} \Sigma \operatorname{Hom}_R(K(x), R)_0 &= \operatorname{Hom}_R(K(x), R)_{0-1} = R \\ \Sigma \operatorname{Hom}_R(K(x), R)_1 &= \operatorname{Hom}_R(K(x), R)_{1-1} = R \\ \Sigma \operatorname{Hom}_R(K(x), R)_i &= 0 \qquad \text{for } i \neq 0, 1. \end{aligned}$$

Thus we have the following complex and map:

$$\begin{array}{llcccc} \Sigma \operatorname{Hom}_R(K(x), R): & 0 \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow 0 \\ \text{degree}: & & 1 & & 0 & \end{array}$$

$$\partial_1^{\Sigma \operatorname{Hom}_R(K(x),R)} = (-1)(-x) = x.$$

The following lemma can be found in [4, (3.29)] described as the covariant Hom and shift functors.

Lemma 5.3. *Let X and Y be R-complexes. Then one has*

$$\operatorname{Hom}_R(X, \Sigma^m Y) \cong \Sigma^m \operatorname{Hom}_R(X, Y).$$

Proof. We proceed by definition showing that the modules and the maps are the same. We begin by considering the left hand side of the desired isomorphism:

$$\operatorname{Hom}_R(X, \Sigma^m Y)_l = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(X_p, (\Sigma^m Y)_{p+l}) = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l-m}).$$

For the right hand side of the desired isomorphism, we have the following:

$$\Sigma^m \operatorname{Hom}_R(X, Y)_l = \operatorname{Hom}_R(X, Y)_{l-m} = \prod_{p \in \mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l-m}).$$

Since they have the same modules, it remains to show that the maps are also the same. For the left hand side, the maps are as follows:

$$\begin{aligned} \partial_l^{\operatorname{Hom}_R(X, \Sigma^m Y)}((\alpha_p)) &= (\partial_{p+l}^{\Sigma^m Y} \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X) \\ &= ((-1)^m \partial_{p+l-m}^Y \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X). \end{aligned}$$

The maps for the right hand side are as follows:

$$\begin{aligned}\partial_l^{\Sigma^m \operatorname{Hom}_R(X,Y)}((\alpha_p)) &= (-1)^m \partial_{l-m}^{\operatorname{Hom}_R(X,Y)}((\alpha_p)) \\ &= (-1)^m (\partial_{p+l-m}^Y \circ \alpha_p - (-1)^{l-m} \alpha_{p-1} \circ \partial_p^X) \\ &= ((-1)^m \partial_{p+l-m}^Y \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X).\end{aligned}$$

Since the maps and the modules are the same, we have the desired isomorphism. □

The following lemma can be found in [4, (3.46)] described as the contravariant Hom and shift functors.

Lemma 5.4. *Let X and Y be R-complexes. Then one has*

$$\operatorname{Hom}_R(\Sigma^{-m} X, Y) \cong \Sigma^m \operatorname{Hom}_R(X, Y).$$

Proof. We proceed by showing that $\operatorname{Hom}_R(\Sigma^{-m} X, Y) \cong \operatorname{Hom}_R(X, \Sigma^m Y)$ and applying Lemma 5.3 to achieve the desired isomorphism. With regard to the modules, we have the following for the right hand side:

$$\begin{aligned}\operatorname{Hom}_R(\Sigma^{-m} X, Y)_l &= \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R((\Sigma^{-m} X)_p, Y_{p+l}) \\ &= \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_{p+m}, Y_{p+l}) \\ &= \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l-m}).\end{aligned}$$

For the left hand side we have the following:

$$\begin{aligned}\operatorname{Hom}_R(X, \Sigma^m Y)_l &= \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, (\Sigma^m Y)_{p+l}) \\ &= \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+l-m}).\end{aligned}$$

Next we want to show that the map

$$\nu : \operatorname{Hom}_R(X, \Sigma^m Y)_l \to \operatorname{Hom}_R(\Sigma^{-m} X, Y)_l$$

is a chain map, giving us the desired isomorphism. Define the map

$$\nu_l : \operatorname{Hom}_R(X, \Sigma^m Y)_l \to \operatorname{Hom}_R(\Sigma^{-m} X, Y)_l$$

by $(\alpha_p) \longmapsto (-1)^{ml}(\alpha_{p+m})$. We need to show

$$\nu_{l-1} \circ \partial_l^{\operatorname{Hom}_R(X,\Sigma^m Y)} = \partial_l^{\operatorname{Hom}_R(\Sigma^{-m} X, Y)} \circ \nu_l.$$

For the left hand side, we have the following:

$$\begin{aligned}\nu_{l-1}(\partial_l^{\operatorname{Hom}_R(X,\Sigma^m Y)}((\alpha_p))) &= \nu_{l-1}((\partial_{p+l}^{\Sigma^m Y} \circ \alpha_p - (-1)^l \alpha_{p-1} \circ \partial_p^X)) \\ &= (-1)^{m(l-1)}(\partial_{p+l+m}^{\Sigma^m Y} \circ \alpha_{p+m} - (-1)^l \alpha_{p+m-1} \circ \partial_{p+m}^X) \\ &= (-1)^{m(l-1)}((-1)^m \partial_{p+m}^Y \circ \alpha_{p+m} \\ &\qquad - (-1)^l \alpha_{p+m-1} \circ \partial_{p+m}^X) \\ &= ((-1)^{m(l-1)+m} \partial_{p+l}^Y \circ \alpha_{p+m} \\ &\qquad - (-1)^{m(l-1)+l} \alpha_{p+m-1} \circ \partial_{p+m}^X) \\ &= ((-1)^{ml} \partial_{p+l}^Y \circ \alpha_{p+m} - (-1)^{ml-m+l} \alpha_{p+m-1} \circ \partial_{p+m}^X).\end{aligned}$$

On the right hand side we have

$$\begin{aligned}\partial_l^{\mathrm{Hom}_R(\Sigma^{-m}X,Y)}(\nu_l((\alpha_p))) &= \partial_l^{\mathrm{Hom}_R(\Sigma^{-m}X,Y)}((-1)^{ml}(\alpha_{p+m}))\\ &= (-1)^{ml}\partial_{p+l}^Y \circ \alpha_{p+m} - (-1)^{ml+l}\alpha_{p+m-1}\circ \partial_p^{\Sigma^{-m}X}\\ &= (-1)^{ml}\partial_{p+l}^Y \circ \alpha_{p+m} - (-1)^{ml+l+m}\alpha_{p+m-1}\circ \partial_{p+m}^{X}.\end{aligned}$$

Since $(-1)^m = (-1)^{-m}$, we have equality and the map ν is a chain map. □

6. Days 6 and 7

Our goal is still to show that the Koszul complex is self-dual. Before we can accomplish this task, we must first discuss the Hom-tensor adjointness map over complexes.

Remark 6.1. Let L, M, and N be R-modules. Hom-tensor adjointness for modules is the map

$$\rho : \mathrm{Hom}_R(L \otimes_R N, M) \to \mathrm{Hom}_R(L, \mathrm{Hom}_R(NM,))$$

given by

$$\rho(\psi)(l)(n) = \psi(l \otimes n)$$

where $l \in L$, $n \in N$, and $\psi \in \mathrm{Hom}_R(L \otimes_R N, M)$. This map is always an isomorphism; see [7, Theorem 2.75, 2.76]

The following lemma can be found in [4, (5.4)] with the proof at [4, (5.9)].

Lemma 6.2. *(Hom-tensor adjointness of complexes)*
Let X, Y, and Z be R-complexes. Then there exists an R-complex isomorphism

$$\rho_{ZYX} : \mathrm{Hom}_R(Z \otimes_R Y, X) \to \mathrm{Hom}_R(Z, \mathrm{Hom}_R(Y, X)).$$

Proof. We use Remark 6.1 to show the existence of isomorphisms between the modules of the two complexes in the above map. We write the lth component of each complex in terms of a module. Modules of the complex $\mathrm{Hom}_R(Z \otimes_R Y, X)$ are as follows:

$$\begin{aligned}\mathrm{Hom}_R(Z \otimes_R Y, X)_l &\cong \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R((Z \otimes_R Y)_n, X_{n+l})\\ &\cong \prod_{n\in\mathbb{Z}} \mathrm{Hom}_R(\coprod_{p\in\mathbb{Z}}(Z_p \otimes_R Y_{n-p}), X_{n+l})\\ &\cong \prod_{p,n\in\mathbb{Z}} \mathrm{Hom}_R(Z_p \otimes_R Y_{n-p}, X_{n+l})\\ &\cong \prod_{p,m\in\mathbb{Z}} \mathrm{Hom}_R(Z_p \otimes_R Y_m, X_{m+p+l}).\end{aligned}$$

The first and second steps are by definition. The third step is standard, and the fourth step follows from setting $n = m + p$.

Modules of the complex $\operatorname{Hom}_R(Z, \operatorname{Hom}_R(Y, X))$ are as follows:

$$\begin{aligned}\operatorname{Hom}_R(Z, \operatorname{Hom}_R(Y, X))_l &\cong \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, \operatorname{Hom}_R(Y, X)_{p+l})\\ &\cong \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, \prod_{m\in\mathbb{Z}} \operatorname{Hom}_R(Y_m, X_{m+p+l}))\\ &\cong \prod_{p,m\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, \operatorname{Hom}_R(Y_m, X_{m+p+l})).\end{aligned}$$

The first and second steps are by definition and the third step is standard.

Remark 6.1 implies that there exists an isomorphism between the individual modules. To show that the isomorphisms between each component yield an isomorphism of complexes, we show that all the diagrams commute.

Set

$$U_l = \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(Z \otimes_R Y_n, X_{n+l}) \cong \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\coprod_{p\in\mathbb{Z}} Z_p \otimes_R X_{n-p}, X_{n+l})$$

and $U'_l = \prod_{n,p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p \otimes_R Y_{n-p}, X_{n+l})$. Then the following diagram commutes

$$\begin{array}{ccc} U_l & \xrightarrow[\cong]{\phi_l} & U'_l \\ {\scriptstyle \partial^U_l}\downarrow & & \downarrow{\scriptstyle \partial^{U'}_l} \\ U_{l-1} & \xrightarrow[\phi_{l-1}]{\cong} & U'_{l-1}. \end{array}$$

We have the following commutative diagram

$$\begin{array}{ccc} Z_p \otimes_R Y_{n-p} & \xrightarrow{\epsilon_p} & \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} \\ & {\scriptstyle \alpha_n\circ\epsilon_p}\searrow & \downarrow{\scriptstyle \alpha_n} \\ & & X_{n+l} \end{array}$$

where ϵ_p is the inclusion map and $\alpha_n \in \operatorname{Hom}_R(\coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p}, X_{n+l})$. For $\beta_{n,p} \in \operatorname{Hom}_R(Z_p \otimes_R Y_{n-p}, X_{n+l})$, we have the following:

$$\begin{aligned}\sum_{p\in\mathbb{Z}} \beta_{n,p} : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} &\longrightarrow X_{n+l}\\ \{z_p \otimes y_{n-p}\}_p &\longmapsto \sum_{p\in\mathbb{Z}} \beta_{n,p}(z_p \otimes y_{n-p}).\end{aligned}$$

The next display defines the maps between U_l and U'_l.

$$\begin{aligned}\phi_l : \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p}, X_{n+l}) &\longrightarrow \prod_{n,p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p \otimes_R Y_{n-p}, X_{n+l})\\ \{\alpha_n : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_n &\longmapsto \{\alpha_n \circ \epsilon_p : Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_{n,p}\\ \psi_l : \prod_{n,p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p \otimes_R Y_{n-p}, X_{n+l}) &\longrightarrow \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p}, X_{n+l})\\ \{\beta_{n,p} : Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_{n,p} &\longmapsto \{\sum_{p\in\mathbb{Z}} \beta_{n,p} : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p}\}_n\end{aligned}$$

If we start with a map in U'_l, then by applying $\phi_l \circ \partial_l^U \circ \psi_l$ we have the following:

$$\psi_l(\{\beta_{n,p} : Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_{n,p})$$
$$= \{\sum_{p\in\mathbb{Z}} \beta_{n,p} : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_n$$
$$\partial_l^U \left(\{\sum_{p\in\mathbb{Z}} \beta_{n,p} : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} \to X_{n+l}\}_n \right)$$
$$= \{\partial_{n+l}^X : \sum_{p\in\mathbb{Z}} \beta_{n,p} - (-1)^l$$
$$\sum_{p\in\mathbb{Z}} \beta_{n+l-1,p} \circ \partial_n^{Z\otimes_R Y} : \coprod_{p\in\mathbb{Z}} Z_p \otimes_R Y_{n-p} \to X_{n+l-1}\}_n$$
$$\phi_l(\{\partial_{n+l}^X \circ \sum_{p\in\mathbb{Z}} \beta_{n,p} - (-1)^l \sum_{p\in\mathbb{Z}} \beta_{n+l-1,p} \circ \partial_n^{Z\otimes_R Y}\}_n)$$
$$= \{\partial_{n+l}^X \sum_{p\in\mathbb{Z}} \beta_{n,p} \circ \epsilon_p - (-1)^l \sum_{p\in\mathbb{Z}} \beta_{n+l-1,p} \circ \partial_n^{Z\otimes_R Y} \circ \epsilon_p$$
$$: Z_p \otimes_R Y_{n-p} \to X_{n+l-1}\}_{n,p}$$

And as an aside we have written:

$$\partial_n^{Z\otimes_R Y}(z_p \otimes y_{n-p}) = \partial_p^Z(z_p) \otimes y_{n-p} + (-1)^p z_p \otimes \partial_{n-p}^Y(y_{n-p})$$
$$\sum_{p\in\mathbb{Z}} \beta_{n+l-1,p}(\partial_n^{Z\otimes_R Y}(\epsilon_p(z_p \otimes y_{n-p}))) =$$
$$\sum_{p\in\mathbb{Z}} \beta_{n+l-1,p}(\cdots, 0, \partial_p^Z(z_p) \otimes y_{n-p}, (-1)^p z_p \otimes \partial_{n-p}^Y(y_{n-p}), 0, \cdots)$$
$$= \beta_{n+l-1,p-1}(\partial_p^Z(z_p) \otimes y_{n-p} + (-1)^p z_p \otimes \partial_{n-p}^Y(y_{n-p}))$$
$$= \left(\beta_{n+l-1,p-1} \circ (\partial_p^Z \otimes_R Y_{n-p}) + (-1)^p \beta_{n+l-1,p-1} \circ (Z_p \otimes_R \partial_{n-p}^Y)\right)(z_p \otimes y_{n-p}).$$

By applying $\partial_l^{U'}$ to a map in U'_l, we have the following:

$$\partial_l^{U'} = \partial_{n+l}^X \beta_{n,p} - \left(-1\right)^l \left(\beta_{n+l-1,p-1} \circ (\partial_p^Z \otimes_R Y_{n-p}) + \right.$$
$$\left. (-1)^{p+l} \beta_{n+l-1,p} \circ (Z_p \otimes_R \partial_{n-p}^Y)\right)(z_p \otimes y_{n-p}).$$

Next we set

$$W_l = \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(Z_p, \prod_{m\in\mathbb{Z}} \mathrm{Hom}_R(Y_m, X_{m+p+l}))$$

and $W'_l = \prod_{m,p\in\mathbb{Z}} \mathrm{Hom}_R(Z_p, \mathrm{Hom}_R(Y_m, X_{m+p+l}))$. Then the following diagram commutes in a manner similar to the previous diagram:

$$\begin{array}{ccc} W_l & \xrightarrow[\cong]{\xi_l} & W'_l \\ \downarrow{\scriptstyle \partial_l^W} & & \downarrow{\scriptstyle \partial_l^{W'}} \\ W_{l-1} & \xrightarrow[\xi_{l-1}]{\cong} & W'_{l-1} \end{array}$$

where we have

$$\partial_l^W(\sigma_{m,p}(z_p)) = \partial_{p+m+l}^X \circ (\sigma_{m,p}(z_p)) - (-1)^{p+l}(\sigma_{m,p}(z_p)) \circ \partial_m^Y - (-1)^l \sigma_{m,p-1}(\partial_p^Z(z_p))$$

for $\sigma_{m,p} \in \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, \prod_{m\in\mathbb{Z}} \operatorname{Hom}_R(X_m, Y_{m+p+l}))$ and $z_p \in Z_p$ for all $p \in \mathbb{Z}$.

Note that each ρ_i is a natural isomorphism – we leave this to the reader to verify. It remains to show that ρ is a chain map by showing that the following diagram commutes:

$$\begin{array}{ccc} W'_l & \xrightarrow{\rho_l} & U'_l \\ \downarrow{\scriptstyle \partial_l^{W'}} & & \downarrow{\scriptstyle \partial_l^{U'}} \\ W'_{l-1} & \xrightarrow[\rho_{l-1}]{} & U'_{l-1}. \end{array}$$

We have the following equation

$$(\rho_{l-1} \circ \partial_l^W)(\{\sigma_{m,p}\}_{m,p}) = \rho_{l-1}(\partial_l^{W'}(\{\sigma_{m,p}\}_{m,p})) : Z_p \otimes_R Y_m \to X_{m+p+l}.$$

For any $z_p \otimes y_m \in Z_p \otimes_R Y_m$ we have

$$\begin{aligned} \rho_{l-1}(\partial_l^{W'}(\{\sigma_{m,p}\}_{m,p}))(z_p \otimes y_m) &= \partial_l^{W'}(\{\sigma_{m,p}\}_{m,p})(z_p)(y_m) \\ &= \partial_{p+l+m}^X((\sigma_{m,p}(z_p)(y_m)) \\ &\quad - (-1)^{p+l}(\sigma_{m,p}(z_p))(\partial_m^Y(y_m)) \\ &\quad - (-1)^l(\sigma_{m,p-1}(\partial_p^Z(z_p)))(y_m). \end{aligned}$$

For the other direction we have

$$\begin{aligned} \partial_l^{U'}(\rho_l(\{\sigma_{m,p}\}_{m,p})) = \partial_{m+p+l}^X &\circ \rho_l(\{\sigma_{m,p}\}_{m,p})_{m+p,p} \\ &- (-1)^l \rho_l(\{\sigma_{m,p}\}_{m,p})_{m,p-1} \circ \partial_p^Y \otimes_R Y_m \\ &- (-1)^{p+l} \rho_l(\{\sigma_{m,p}\}_{m,p})_{m-1,p} \circ Z_p \otimes_R \partial_m^Y. \end{aligned}$$

Evaluated at $z_p \otimes y_m \in Z_p \otimes_R Y_m$ we have

$$\begin{aligned} \partial_{m+p+l}^X((\sigma_{m,p}(z_p)(y_m)) &- (-1)^l(\sigma_{m,p-1}(\partial_p^Z(z_p)))(y_m) \\ &- (-1)^{p+l}(\sigma_{m,p}(z_p))(\partial_m^Y(y_m)). \end{aligned}$$

Therefore the diagram commutes as desired. □

7. Day 7

The following theorem can be found in [2, Proposition 1.6.10(b)].

Theorem 7.1. *$K(\underline{x})$ is self-dual, that is, $\Sigma^n \operatorname{Hom}_R(K(\underline{x}), R) \cong K(\underline{x})$ where $\underline{x} = x_1, \cdots, x_n$.*

Proof. We proceed by induction on n. The base case is from Remark 6.1.

Inductive Step: Assume that the theorem holds for the case $n-1$. The inductive hypothesis implies the first of the next two isomorphisms

$$\begin{aligned} K(x_1, \cdots, x_{n-1}) &\cong \Sigma^{n-1} \operatorname{Hom}_R(K(x_1, \cdots, x_{n-1}), R) \\ \operatorname{Hom}_R(K(x_n), R) &\cong \Sigma^{-1} K(x_n). \end{aligned}$$

The second isomorphism follows from the base case. These isomorphisms and Hom-tensor adjointness explain the second step in the next display:

$$\begin{aligned}\Sigma^n \operatorname{Hom}_R(K(x_1,\cdots,x_n),R) &\cong \Sigma^n \operatorname{Hom}_R(K(x_1,\cdots,x_{n-1})\otimes_R K(x_n),R)\\ &\cong \Sigma^n \operatorname{Hom}_R(\Sigma^{n-1}\operatorname{Hom}_R(K(x_1,\cdots,x_{n-1}),R),\Sigma^{-1}K(x_n))\\ &\cong \Sigma^n\Sigma^{1-n}\Sigma^{-1}\operatorname{Hom}_R(\operatorname{Hom}_R(K(x_1,\cdots,x_{n-1}),R),K(x_n))\\ &\cong \operatorname{Hom}_R(\operatorname{Hom}_R(K(x_1,\cdots,x_{n-1}),R),K(x_n))\\ &\cong K(x_1,\cdots,x_{n-1})\otimes_R \operatorname{Hom}_R(R,K(x_n))\\ &\cong K(x_1,\cdots,x_n).\end{aligned}$$

The first step is by definition of the Koszul complex and the third step is from Lemmas 5.3 and 5.4. The fourth step follows from definition of shift and the sixth step is tensor cancellation together with the definition of the Koszul complex. The fifth step follows from Hom-evaluation, which is proved in the next theorem. □

Remark 7.2. Let X, Y, and Z be R-modules. The *Hom-evaluation* homomorphism for modules is the map

$$\theta_{XYZ}\colon X\otimes_R \operatorname{Hom}_R(Y,Z)\to \operatorname{Hom}_R(\operatorname{Hom}_R(X,Y),Z)$$

defined by $\theta_{XYZ}(x\otimes\psi)(\phi)=\psi(\phi(x))$. This map is an isomorphism when X is finite and projective or when X is finite and Z is injective; we only need the first condition, see [7, Lemma 3.55].

The following theorem can be found in [4, (5.6)] with the proof at [4, (5.11)].

Theorem 7.3. *Let X, Y, and Z be R-complexes. If Z is a bounded complex of finitely generated projective R-modules, then there exists an isomorphism of complexes*

$$Z\otimes_R \operatorname{Hom}_R(Y,X)\cong \operatorname{Hom}_R(\operatorname{Hom}_R(Z,Y),X).$$

Proof. We need to show that both complexes have the same modules and that the differentials match up. We will show the first and leave the second for the interested reader.

Set $U=Z\otimes_R \operatorname{Hom}_R(Y,X)$ and $W=\operatorname{Hom}_R(\operatorname{Hom}_R(Z,Y),X)$. Then in the lth index of U we have the following modules:

$$\begin{aligned}U_l &= \coprod_{p\in\mathbb{Z}} Z_p\otimes_R \operatorname{Hom}_R(Y,X)_{l-p}\\ &= \coprod_{p\in\mathbb{Z}} Z_p\otimes_R \prod_{m\in\mathbb{Z}}\operatorname{Hom}_R(Y_m,X_{m+l-p})\\ &= \prod_{p\in\mathbb{Z}} Z_p\otimes_R \prod_{m\in\mathbb{Z}}\operatorname{Hom}_R(Y_m,X_{m+l-p})\\ &= \prod_{p,m\in\mathbb{Z}} Z_p\otimes_R \operatorname{Hom}_R(Y_m,X_{m+l-p}).\end{aligned}$$

In the lth index of W we have the following modules:

$$\begin{aligned} W_l &= \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\operatorname{Hom}_R(Z,Y)_n, X_{n+l}) \\ &= \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, Y_{p+n}), X_{n+l}) \\ &= \prod_{n\in\mathbb{Z}} \operatorname{Hom}_R(\coprod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Z_p, Y_{p+n}), X_{n+l}) \\ &= \prod_{n,p\in\mathbb{Z}} \operatorname{Hom}_R(\operatorname{Hom}_R(Z_p, Y_{p+n}), X_{n+l}) \\ &\cong \prod_{n,p\in\mathbb{Z}} Z_p \otimes_R \operatorname{Hom}_R(Y_{p+n}, X_{n+l}). \end{aligned}$$

The third step follows from the fact that Z_p is bounded and the last step follows from Remark 7.2 because Z_p consists of finitely generated projective modules. The modules are the same by setting $p+n=m$. □

8. Day 8

The goal for today is to show that $(\underline{x})\operatorname{H}_i(K(\underline{x})) = 0$ for each i where $\underline{x} = x_1, x_2, \cdots, x_d \in R$ and $K(\underline{x})$ is the Koszul complex. First, we need a few lemmas concerning null-homotopic morphisms; see [4, (1.45)] for defintion.

Lemma 8.1. *Let X, Y, and Z be complexes and let $\alpha : X \to Y$ be a chain map. If $\alpha \sim 0$, then $\alpha \otimes_R Z \sim 0$.*

Proof. It is left to the reader to show that $\alpha \otimes_R Z$ is a chain map. Recall that $\alpha\otimes_R Z : X\otimes_R Z \to Y\otimes_R Z$ can be written as $\coprod_{p+q=i} X_p\otimes_R Z_q \to \coprod_{p+q=i} Y_p\otimes_R Z_q$ where in each coordinate we have the map

$$\begin{aligned} X_p \otimes_R Z_q &\to Y_p \otimes_R Z_q \\ x_p \otimes z_q &\mapsto \alpha(x_p) \otimes z_q. \end{aligned}$$

Since $\alpha \sim 0$, there exists $S_i : X_i \to Y_{i+1}$ with $\alpha_i = S_{i-1} \circ \partial_i^X + \partial_{i+1}^Y \circ S_i$. Let $t_i : (X \otimes_R Z)_i \to (Y \otimes_R Z)_{i+1}$, otherwise written as $t_i : \coprod_{p+=q=i} X_p \otimes_R Z_q \to \coprod_{p+q=i+1} Y_p \otimes_R Z_q$, be defined in each coordinate as follows:

$$\begin{aligned} X_p \otimes_R Z_q &\to Y_{p+1} \otimes_R Z_q \\ x_p \otimes z_q &\mapsto S_p(x_p) \otimes z_q. \end{aligned}$$

To show that $t_{i-1} \circ \partial_i^{X \otimes_R Z} + \partial_{i+1}^{Y \otimes_R Z} \circ t_i = (\alpha \otimes_R Z)_i$ we have the following computations:

$$\begin{aligned}
(t_{i-1} \circ \partial_i^{X \otimes_R Z} &+ \partial_{i+1}^{Y \otimes_R Z} \circ t_i)(x_p \otimes z_q) \\
&= t_{i-1}(\partial_i^{X \otimes_R Z}(x_p \otimes z_q) + \partial_{i+1}^{Y \otimes_R Z}(t_i(x_p \otimes z_q))) \\
&= t_{i-1}(\partial_p^X(x_p) \otimes z_q + (-1)^p x_p \otimes \partial_q^Z(z_q)) + \partial_{i-1}^{Y \otimes_R Z}(S_p(x_p) \otimes z_q) \\
&= S_{p-1}(\partial_p^X(x_p)) \otimes z_q + (-1)^p S_p(x_p) \otimes \partial_q^Z(z_q) \\
&\quad + \partial_{p+1}^Y(S_p(x_p)) \otimes z_q + (-1)^{p+1} S_p(x_p) \otimes \partial_q^Z(z_q) \\
&= S_{p-1}(\partial_p^X(x_p)) \otimes z_q + \partial_{p+1}^Y(S_p(x_p)) \otimes z_q \\
&= [S_{p-1}(\partial_p^X(x_p)) + \partial_{p+1}^Y(S_p(x_p))] \otimes z_q \\
&= \alpha_p(x_p) \otimes z_q \\
&= (\alpha \otimes_R Z)_i(x_p \otimes z_q).
\end{aligned}$$

□

Lemma 8.2. *Let X and Z be complexes and let $\mu_X^t : X \to X$ given by $x \mapsto xt$ be a homothety map. Then we have the following equalities*
(a) $\mu_X^t \otimes_R Z = \mu_{X \otimes_R Z}^t$;
(b) $\operatorname{Hom}_R(\mu_X^t, Z) = \mu_{\operatorname{Hom}_R(X,Z)}^t$; *and*
(c) $\operatorname{Hom}_R(Z, \mu_X^t) = \mu_{\operatorname{Hom}_R(Z,X)}^t$.

Proof. We will show the proof for part (a) and leave the rest as an exercise for the reader.

We have the following computation:

$$\begin{aligned}
(\mu_X^t \otimes_R Z)_i(x_p \otimes z_q) &= (\mu_X^t)_p(x_p) \otimes_R z_q \\
&= (tx_p) \otimes z_q \\
&= t(x_p \otimes z_q) \\
&= \mu_{X \otimes_R Z}^t(x_p \otimes z_q).
\end{aligned}$$

□

Lemma 8.3. *Let X and Y be complexes and let $\alpha : X \to Y$ be a chain map. If $\alpha \sim 0$, then $\mathrm{H}_i(\alpha) = 0$ for all i.*

Proof. Since $\alpha \sim 0$, we have $\alpha_i = S_{i-1} \circ \partial_i^X + \partial_{i+1}^Y \circ S_i$. Then $\mathrm{H}_i(\alpha) : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$. Let $\{\beta\} \in \mathrm{H}_i(X)$. Then $\beta \in \operatorname{Ker}(\partial_i^X)$ and this explains the third step in the following sequence

$$\begin{aligned}
\mathrm{H}_i(\alpha)(\{\beta\}) &= \{\alpha(\beta)\} \\
&= \{S_{i-1}(\partial_i^X(\beta)) + \partial_{i+1}^Y(S_i(\beta)\} \\
&= \{\partial_{i+1}^Y(S_i(\beta))\} \\
&= 0.
\end{aligned}$$

The fourth step follows from $\partial_{i+1}^Y(S_i(\beta)) \in \operatorname{Im}(\partial_{i+1}^Y)$. □

Corollary 8.4. *Let X and Y be complexes and let $\alpha, \alpha' : X \to Y$ be chain maps. If $\alpha \sim \alpha'$, then $\mathrm{H}_i(\alpha) = \mathrm{H}_i(\alpha')$ for all i.*

Proof. Since $\alpha \sim \alpha'$, we have $\alpha - \alpha' = 0$. This implies that $\mathrm{H}_i(\alpha - \alpha') = 0$ for all i. Thus $\mathrm{H}_i(\alpha) - \mathrm{H}_i(\alpha') = 0$. Hence we conclude that $\mathrm{H}_i(\alpha) = \mathrm{H}_i(\alpha')$. □

Lemma 8.5. *Let X be a complex and let $\mu_X^t : X \to X$ be a homothety map. Then* $\mathrm{H}_i(\mu_X^t) = \mu^t_{\mathrm{H}_i(x)}$.

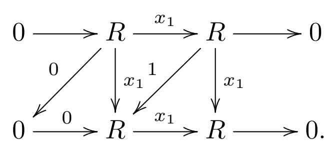
Proof. Let $\{\beta\} \in \mathrm{H}_i(X)$. Then we have the following computations:

$$\begin{aligned} \mathrm{H}_i(\mu_X^t)(\{\beta\}) &= \{\mu_X^t(\beta)\} \\ &= \{t\beta\} \\ &= t\{\beta\} \\ &= \mu^t_{\mathrm{H}_i(X)}(\{\beta\}). \end{aligned}$$

□

Proposition 8.6. *Let $\underline{x} = x_1, x_2, \cdots, x_d \in R$ and let $K(\underline{x})$ be the Koszul complex. Then $(\underline{x})\,\mathrm{H}_i(K(\underline{x})) = 0$ for each i.*

Proof. It suffices to show that $x_1 \mathrm{H}_i(K(x_1) \otimes_R L) = 0$. First we show that we have $\mu^{x_1}_{K(x_1)} \sim 0$ by the following diagram



Since $\mu^{x_1}_{K(x_1)} \sim 0$, Lemma 8.1 implies that $\mu^{x_1}_{K(x_1)} \otimes_R L \sim 0$. Then by Lemma 8.2 we have $\mu^{x_1}_{K(x_1)} \otimes_R L = \mu^{x_1}_{K(x_1)\otimes_R L}$. Thus $\mu^{x_1}_{K(x_1)\otimes_R L} \sim 0$. Lemma 8.3 implies that $\mathrm{H}_i(\mu^{x_1}_{K(x_1)\otimes_R L}) = 0$ for all i. Then by Lemma 8.5 we have $\mathrm{H}_i(\mu^{x_1}_{K(x_1)\otimes_R L}) = \mu^{x_1}_{\mathrm{H}_i(K(x_1)\otimes_R L)}$. Therefore we conclude that $x_1 \mathrm{H}_i(K(x_1) \otimes_R L) = 0$. □

9. Day 9

We now discuss depth sensitivity. For definitions pertaining to regular and weakly regular sequences, see [2, Definition 1.1.1]. The first part of the next theorem is also proved in [2, Theorem 1.6.16]

Theorem 9.1. *Let M be a non-zero R-module and let $\underline{x} = x_1, \cdots, x_d$ be elements of R.*

(i) *If $(\underline{x})$ contains a weakly M-regular sequence of length t, then*
$$\mathrm{H}_{j-d}(K(\underline{x}; M)) = 0$$
for all $j < t$.

(ii) *If M is finitely generated and $\underline{x}M \neq M$, then we have* $\operatorname{depth}_R(K(\underline{x}; M)) =$ *min*$\{j \mid \mathrm{H}_{j-d}(K(\underline{x}; M)) \neq 0\}$.

Proof. Part (i): Suppose $y \in (\underline{x})$ is a non-zero divisor on M. Then the following sequence is exact:
$$0 \to M \xrightarrow{y} M \to M/yM \to 0.$$
We tensor this with $K(\underline{x}; R)$ to obtain the exact sequence
$$0 \to K(\underline{x}; M) \xrightarrow{y} K(\underline{x}; M) \to K(\underline{x}; M/yM) \to 0.$$

This yields the following long exact sequence:

$$\cdots \to \mathrm{H}_j(K(\underline{x};M)) \xrightarrow{y} \mathrm{H}_j(K(\underline{x};M)) \to \mathrm{H}_j(K(\underline{x};M/yM)) \to \mathrm{H}_{j+1}(K(\underline{x};M)) \to \cdots.$$

Since $y \in (\underline{x})$, by Proposition 8.6 we have $(\underline{x})\,\mathrm{H}_j(K(\underline{x};M)) = 0$. Thus $y = 0$ as a map. Then we have the exact sequence

$$0 \to \mathrm{H}_j(K(\underline{x};M)) \to \mathrm{H}_j(K(\underline{x};M/yM)) \to \mathrm{H}_{j+1}(K(\underline{x};M)) \to 0.$$

We proceed by induction on t, the length of the weakly M-regular sequence.

Base case: $t = 0$. We have the empty sequence and show $\mathrm{H}_{j-d}(K(\underline{x};M)) = 0$. For $j < 0$, we have the sequence

$$0 \to K^{-d} \to K^{-d+1} \to \cdots \to K^{-1} \to K^0 \to 0.$$

Since $j < 0$, we have $j - d < -d$. Thus $K^{j-d}(\underline{x};M) = 0$. Hence we conclude that $\mathrm{H}_{j-d}(K(\underline{x};M)) = 0$.

Inductive step. Suppose that $y_1, \cdots, y_{t-1} \in (\underline{x})$ is a weakly M-regular sequence. Then $\mathrm{H}_{j-d}(K(\underline{x};M)) = 0$ for $j < t-1$. Let $y_1, \cdots, y_t \in (\underline{x})$ be a weakly M-regular sequence. Then $y_2, \cdots, y_t \in (\underline{x})$ is a weakly M/y_1M-regular sequence. The inductive hypothesis applied to M/y_1M implies that $\mathrm{H}_{j-d}(K(\underline{x};M/y_1M)) = 0$ for $j < t-1$. Consider the exact sequence

$$0 \to \mathrm{H}_{j-1-d}(K(\underline{x};M)) \to \mathrm{H}_{j-1-d}(K(\underline{x};M/y_1M)) \to \mathrm{H}_{j-d}(K(\underline{x};M)) \to 0.$$

Since the middle term is zero, we have that $\mathrm{H}_{j-d}(K(\underline{x};M)) = 0$.

Part (ii): Set $t = \mathrm{depth}_R(K(\underline{x};M))$ and note that since $(\underline{x})M \neq M$, we have that $(\underline{x})$ is an M-regular sequence. By part (i), it suffices to show $\mathrm{H}_{t-d}(K(\underline{x};M)) \neq 0$. We proceed by induction.

Base case: assume that $t = 0$. This implies that $\mathrm{depth}_R(K(\underline{x},M)) = 0$. Thus the maximal M-regular sequence is the empty sequence. This implies that there are only zero divisors in $(\underline{x})$. Hence $(\underline{x}) \subseteq \mathfrak{p}$ for some associated prime $\mathfrak{p}$ of M. We have $\mathrm{depth}_R(K(\underline{x};M)) = 0$ if and only if $(\underline{x})$ is contained in $\cup_{\mathfrak{p}\in \mathrm{Ass}_R(M)}\mathfrak{p}$. By the prime avoidance theorem, there exists a $\mathfrak{p} \in \mathrm{Ass}_R(M)$ such that $(\underline{x}) \subseteq \mathfrak{p}$. By definition there exists a non-zero element $m \in M$ such that $(0 : m) = \mathfrak{p}$. In particular, $m(\underline{x}) = 0$.

From the exact sequence

$$0 \to M^{-d} \xrightarrow{\begin{pmatrix} \pm x_d \\ \pm x_{d-1} \\ \vdots \\ \pm x_1 \end{pmatrix}} M^{-d+1} \to \cdots \to M^{-1} \to M \to 0$$

we have $m \in \mathrm{Ker}\,\partial^M_{-d} = \mathrm{H}_{-d}(K(\underline{x};M))$. This implies that

$$m \mapsto \begin{pmatrix} \pm m x_d \\ \pm m x_{d-1} \\ \vdots \\ \pm m x_1 \end{pmatrix} = 0.$$

Thus $mx_i = 0$ for $i = 1, \cdots, d$. Since $m(\underline{x}) = 0$, we have $(0 :_M \underline{x}) = \mathrm{H}_{-d}(K(\underline{x};M))$. Therefore we have a non-zero element $m \in \mathrm{H}_{-d}(K(\underline{x};M))$. This implies that $\mathrm{H}_{-d}(K(\underline{x};M)) \neq 0$.

Inductive step. Assume that $t \geqslant 1$ and that we have $\operatorname{depth}_R(K(\underline{x};M)) = min\{j \mid \mathrm{H}_{j-d}(K(\underline{x};M)) \neq 0\}$ for all $\operatorname{depth}_R(K(\underline{x};M)) < t$. Let $y_1, \cdots, y_t$ be an M-regular sequence in $(\underline{x})$. Note that $\operatorname{depth}_R(K(\underline{x};M/y_1M)) = t-1$ and consider the exact sequence

$$0 \to \mathrm{H}_{t-1-d}(K(\underline{x};M)) \to \mathrm{H}_{t-1-d}(K(\underline{x};M/y_1M)) \to \mathrm{H}_{t-d}(K(\underline{x};M)).$$

Part (i) implies that the first term is zero. Since the second term is non-zero, we have that the last term of the sequence is non-zero. □

Remark 9.2. Ext is depth sensitive. In the case where $IM \neq M$ and M is finitely generated, the module $\operatorname{Ext}_R^{\operatorname{depth}(I,M)}(R/I, M)$ is non-zero while the modules $\operatorname{Ext}_R^0(R/I,M), \operatorname{Ext}_R^1(R/I,M), \ldots, \operatorname{Ext}_R^{\operatorname{depth}(I,M)-1}(R/I,M)$ are all zero.

When $\underline{x} = x_1, \cdots, x_d$, the homologies $\mathrm{H}_d(K(\underline{x};M))$ and $\mathrm{H}_{d-1}(K(\underline{x};M))$ are both zero. The first non-zero homology is $\mathrm{H}_{d-\operatorname{depth}_R(K(\underline{x};M))}(K(\underline{x};M))$. For example, if for all $i > 0$, we have $\mathrm{H}_i(K(\underline{x};M)) = 0$, then $d - \operatorname{depth}_R(K(\underline{x};M)) = 0$ implying that $d = \operatorname{depth}_R(K(\underline{x};M))$. Thus $\underline{x}$ is M-regular. Hence we conclude that $\mathrm{H}_i(K(\underline{x};M)) = 0$ for all $i > 0$ if and only if $\underline{x}$ is M-regular in the case that M is finitely generated and $(\underline{x})M \neq M$.

10. Day 10

Today we get the motivational speech. We will discuss two nice properties. Although not all of the following requires R to be local, we will impose that condition nonetheless to simplify things.

Assume $(R, \mathfrak{m}, k)$ is local and $\mathfrak{m} = (\underline{x})R$ where $\underline{x} = x_1, \cdots, x_n$ is a minimal generating sequence for $\mathfrak{m}$.

Q: What is so great about Koszul complexes?

A: The Koszul complex is a tool for proving theorems. If we were to be glib, we could call it 'applied' math.

Property 10.1. (a) If X is an R-complex such that $\mathrm{H}_i(X)$ is finitely generated for all i and $\mathrm{H}_i(X) = 0$ for all $i \ll 0$, then X is exact if and only if $K \otimes_R X$ is exact. This is a version of "K is faithfully flat".

(b) If $f : X \to Y$ is a chain map over R such that $\mathrm{H}_i(X)$ and $\mathrm{H}_i(Y)$ are finitely generated for all i and $\mathrm{H}_i(X) = 0 = \mathrm{H}_i(Y)$ for all $i \ll 0$, then f is a quasiisomorphism if and only if $K \otimes_R f$ is a quasiisomorphism. Here we apply part (a) to $\operatorname{Cone}(f)$.

The slogan for the next property is that 'K is ringy enough to do stuff with'. The next property concerns DG algebras; see Definition 11.3.

Property 10.2. K is a DG-algebra over R as well as commutative, local, bounded, and free over R. In practice, this means that K is a good substitute for $R/(\underline{x})$ when $\underline{x}$ is not R-regular.

The next example shows how useful these two properties are. Some details of the example are glossed over because of the complexity involved- this should not detract from the gist of the example, which is, Koszul complexes are extremely useful.

Example 10.3. How to prove that R has a finite number of semi-dualizing modules (up to isomorphism classes)?

Case 1. If R is Cohen-Macaulay, let $\underline{y} = y_1, \cdots, y_d \in \mathfrak{m}$ be a maximal R-regular sequence. Let $R \to R/(\underline{y}) = \overline{R}$. Then $\mathrm{pd}_R(\overline{R}) = d < \infty$ and $\overline{R}$ is artinian. Thus $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(\overline{R})$ where $\mathfrak{S}_0(R)$ denotes the set of isomorphism classes of semidualizing R-modules. Therefore it suffices to show that $|\mathfrak{S}_0(\overline{R})| < \infty$.

If R contains a field, then using representation theory and work we can show that $|\mathfrak{S}_0(\overline{R})| < \infty$. The work required in representation theory depends crucially on the fact that $\overline{R}$ if of finite length and contains a field.

Case 2. the general case. We use K in place of $\overline{R}$.

Step 1. K is a DG-algebra (in the sense that it is a ring that contains a field) over R. The D stands for differential and the G stands for graded. Define multiplication on K by

$$K_i \otimes_R K_j \xrightarrow{\mu_{ij}} K_{i+j}$$
$$(e_{s_1} \wedge \cdots \wedge e_{s_i}) \otimes (e_{t_1} \wedge \cdots \wedge e_{t_j}) \mapsto e_{s_1} \wedge \cdots \wedge e_{s_i} \wedge e_{t_1} \wedge \cdots \wedge e_{t_j}.$$

Recall that $e_i \wedge e_i = 0$ and $e_i \wedge e_j = -e_j \wedge e_i$. Therefore any repetitions cancel and we can permute the elements at will into ascending order with a -1 raised to the number of permutations performed.

This is graded commutative, that is, $x_i x_j = (-1)^{ij} x_j x_i$. This is associative, and as distributive as it can be in the sense that we cannot add arbitrary elements. We cannot add elements of different degrees, but we do have the following $a_i(b_j + c_j) = a_i b_j + a_i c_j$. This also adheres to the Leibniz rule, that is

$$\partial(x_i x_j) = \partial(x_i) x_j + (-1)^i x_i \partial(x_j).$$

Step 2. There exists a DG-algebra homomorphism $\phi : R \to K$ defined as follows

$$\begin{array}{ccccccccccc} & & & & & & 0 & \longrightarrow & R & \longrightarrow & 0 \\ & & & & & & & & \big\downarrow{\scriptstyle id} & & \\ 0 & \longrightarrow & K_e & \longrightarrow & \cdots & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0. \end{array}$$

This map respects addition, multiplication, and differentials. The map may not be reversed to go up because that map does not respect differentials.

Step 3. K is finitely generated and free over R. Therefore K is flat. In fact, by Property 10.1 it is in a sense faithfully flat.

Step 4. If M is an R-module, then $K \otimes_R M$ is a DG-module over K. The scalar multiplication is defined as follows:

$$K_i \otimes_R (K \otimes_R M)_j \to (K \otimes_R M)_{i+j}$$
$$x_i \otimes (x_j \otimes m) \mapsto (x_i x_j) \otimes m.$$

This is associative, distributive, and satisfies a version of the Leibniz rule.

Step 5. If M is an R-complex, then $K \otimes_R M$ is a DG-module over K.

Step 6. If C is a semidualizing R-module, then $K \otimes_R C$ is a semidualizing DG-module over K.

Step 7. Step 6 implies that we need Ext. That is, we need $\mathrm{Hom}_R(-,-)$ and resolutions. This is not quite enough, as what we really need is $\mathbf{R}\mathrm{Hom}_R(-,-)$, but because this goes into much depth we will waive our hands a bit at this point. If for R we have $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \neq 0$ and $R \xrightarrow{\cong} \mathrm{Hom}_R(C,C)$, then for K we have $K \xrightarrow{\simeq} \mathrm{Hom}_K(P,P)$ where P is a semi-projective resolution of $K \otimes_R C$. All we really need for R is to show that $R \xrightarrow{\simeq} \mathrm{Hom}_R(P, P)$ where P is a projective resolution of C.

Step 8. We have $\mathfrak{S}_0(R) \hookrightarrow \mathfrak{S}_0(K)$ from the faithfully flat conditions. We want to show that $\mathfrak{S}_0(K)$ is finite. The homologies $\mathrm{H}_i(K)$ are finite dimensional vector spaces over k. Thus we have $K \simeq F$ as a DG-algebra where F is a finite dimensional vector space over k (that is, a finite length ring containing a field) and F is a DG-algebra. Since a quasiisomorphism of DG-algebras is just as good as an isomorphism, we have $\mathfrak{S}_0(K) \approx \mathfrak{S}_0(F)$.

Next we reprove the representation theory results from Case 1 for the DG setting. This is nontrivial.

Disadvantage of Koszul complexes: It is very technical stuff.

Advantage of Koszul complexes: It allows us to prove theorems that others cannot.

11. Day 11

Today we introduce DG algebras. First we provide some background information and review.

Basic Constructions.

Remark 11.1. (i) All rings are assumed to be commutative.

(ii) Let R be a ring. A complex of R-modules is a sequence

$$F = \cdots \to F_{n+1} \xrightarrow{\partial^F_{n+1}} F_n \xrightarrow{\partial^F_n} F_{n-1} \to \cdots$$

such that $\partial^F_n \circ \partial^F_{n+1} = 0$ for all $n \in \mathbb{Z}$.

(iii) The underlying R-module $\{F_n\}_{n\in\mathbb{Z}}$ is denoted $F^\sharp$. Existing literature may sometimes use $F^\natural$. The meaning behind this is that we are removing the differentials, hence viewing F in the so called 'nude'.

(iv) We write $|x|$ to denote the degree of an element x. That is, if $|x| = n$ then $x \in F_n$.

Remark 11.2. Let E, F, and G be complexes of R-modules. Recall:

(i) A degree d homomorphism $\beta : F \to G$ is the collection of R-linear maps $\{\beta_n : F_n \to G_{n+d}\}_{n\in\mathbb{Z}}$. All degree d homomorphisms from F to G form an R-module $\mathrm{Hom}_R(F, G)_d$. This is the degree d component of $\mathrm{Hom}_R(F, G)$ in which the boundary on β is defined by

$$\partial(\beta) = \partial^G \circ \beta - (-1)^d \beta \circ \partial^F$$

where $|\beta| = d$.

(ii) The tensor product $E \otimes_R F$ consists of $(E \otimes_R F)_n = \oplus_{i+j=n} E_i \otimes_R F_j$ and the maps $\partial^{E\otimes_R F}_{|e|+|f|}(e \otimes f) = \partial^E_{|e|}(e) \otimes f + (-1)^{|e|} e \otimes \partial^F_{|f|}(f)$.

DG Algebras, DG modules, and DG homomorphisms. We will work over a commutative local ring $(R, \mathfrak{m})$.

Definition 11.3. A *DG algebra* is an R-complex (A, ∂^A) with $1 \in A_0$ (unit) and a morphism of complexes called the *product*

$$\mu^A : A \otimes_R A \to A$$
$$a \otimes b \mapsto ab$$

with the following properties:

(i) unitary: $1a = a = a1$ for all $a \in A$; and
(ii) asociative: $(ab)c = a(bc)$ for all $a, b, c \in A$.

In addition, we assume that A is commutative if the following hold:

(iii) $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$ and $a^2 = 0$ when $|a|$ is odd; and
(iv) $A_i = 0$ for all $i < 0$.

Note that the distributive laws are automatic. When parts (iii) and (iv) are not satisfied, we speak of "associative DG algebras".

Fact 11.4. For all $a \in A_i$ and $b \in A_j$ we have the equality

$$\partial^A_{i+j}(ab) = \partial^A_i(a)b + (-1)^i a\partial^A_j(b) \tag{11.4.1}$$

called the *Leibniz rule*. We verify that this is a chain map using the following commutative diagram:

$$\begin{array}{ccc}
(A \otimes_R A)_{i+j} & \xrightarrow{\mu_{i+j}} & A_{i+j} \\
\downarrow{\scriptstyle \partial^{A\otimes_R A}_{i+j}} & & \downarrow{\scriptstyle \partial^A_{i+j}} \\
(A \otimes_R A)_{i+j-1} & \xrightarrow[\mu_{i+j-1}]{} & A_{i+j-1}.
\end{array}$$

The computations are as follows:

$$\begin{aligned}
\partial^A_{i+j}(ab) &= \partial^A_{i+j}(\mu_{i+j}(a \otimes b)) \\
&= \mu_{i+j-1}(\partial^{A\otimes_R A}_{i+j}(a \otimes b)) \\
&= \mu_{i+j-1}(\partial^A_i(a) \otimes b + (-1)^i a \otimes \partial^A_j(b)) \\
&= \partial^A_i(a)b + (-1)^i a\partial^A_j(b).
\end{aligned}$$

Example 11.5. (i) We regard R as a DG algebra concentrated in degree 0.
(ii) The Koszul complex $K = K(a_1, \cdots, a_n)$ where $a_1, \cdots, a_n \in \mathfrak{m}$ can be realized as an exterior algebra and the wedge product endows it with a DG algebra structure which is commutative with multiplication defined as follows

$$\begin{aligned}
\mu : K \otimes_R K &\to K \\
a \otimes b &\mapsto a \wedge b.
\end{aligned}$$

Recall that for the wedge product we have $e_i \wedge e_j = \begin{cases} 0, & \text{if } i = j \\ e_i \wedge e_j & \text{if } i < j \\ -e_j \wedge e_i & \text{if } i > j \end{cases}$ implying that $a \wedge b = (-1)^{|a||b|} b \wedge a$.

It remains to verify the Leibnez rule. Let $a \in K_i$ and $b \in K_j$. We show that

$$\partial^K_{i+j}(ab) = \partial^K_i(a)b + (-1)^i a\partial^K_j(b).$$

Suppose that $a = e_{t_1} \wedge \cdots \wedge e_{t_i}$ and $b = e_{f_1} \wedge \cdots \wedge e_{f_j}$ are strictly ascending. We have the following computations:

$$\begin{aligned}
\text{LHS} &= \partial^K_{i+j}((e_{t_1} \wedge \cdots \wedge e_{t_i}) \wedge (e_{f_1} \wedge \cdots \wedge e_{f_j})) \\
&= \sum_{l=1}^{i} (-1)^{l+1} x_{t_l} e_{t_1} \wedge \cdots \wedge \widehat{e}_{t_l} \wedge \cdots \wedge e_{t_i} \wedge e_{f_1} \wedge \cdots \wedge e_{f_j} \\
&\quad + \sum_{l=1}^{j} (-1)^{l+i+1} x_{f_l} e_{t_1} \wedge \cdots \wedge e_{t_i} \wedge e_{f_1} \wedge \cdots \wedge \widehat{e}_{f_l} \wedge \cdots \wedge e_{f_j} \\
\text{RHS} &= (\sum_{l=1}^{i} (-1)^{l+1} x_{t_l} e_{t_1} \wedge \cdots \wedge \widehat{e}_{t_l} \wedge \cdots \wedge e_{t_i}) \wedge (e_{f_1} \wedge \cdots \wedge e_{f_j}) \\
&\quad + (-1)^i (e_{t_1} \wedge \cdots \wedge e_{t_i}) \wedge (\sum_{l=1}^{j} (-1)^{l+1} x_{f_l} e_{f_1} \wedge \cdots \wedge \widehat{e}_{f_l} \wedge \cdots \wedge e_{f_j}).
\end{aligned}$$

We see that the left hand side is equal to the right hand side. If changes to the order need to be made, the same changes will be made on both sides.

Definition 11.6. A *morphism of DG algebras* is a morphism of complexes

$$\phi : A \to A'$$

such that

(i) $\phi(1) = 1$; and
(ii) $\phi(ab) = \phi(a)\phi(b)$ for all $a, b \in A$.

If this is the case, we say that A' is a DG-algebra over A.

Example 11.7. Let A be a DG algebra. Then the map $f : R \to A$ is a morphism from R to the DG algebra A. The morphism of complexes is as described below

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & R & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \\
\cdots & \longrightarrow & A_2 & \longrightarrow & A_1 & \longrightarrow & A_0 & \longrightarrow & 0
\end{array}$$

with $f(1_R) = 1_A$.

12. Day 12

Proposition 12.1. *If A and A' are DG R-algebras, then the tenor product of complexes $A \otimes_R A'$ is a DG R-algebra with the map*

$$\begin{aligned}
\mu^{A \otimes_R A'} : (A \otimes_R A') \otimes_R (A \otimes_R A') &\longrightarrow A \otimes_R A' \\
(a \otimes a') \otimes (b \otimes b') &\longmapsto (-1)^{|a'||b|} ab \otimes a'b'.
\end{aligned}$$

Proof. The unit of $A \otimes_R A'$ is $1 \otimes 1$ where the degree of 1 is zero. To show that $A \otimes_R A'$ satisfies associativity, let $a, b, c \in A$ and $a', b', c' \in A'$. Then we have the following computations to show that $((a\otimes a')(b\otimes b'))(c\otimes c') = (a\otimes a')((b\otimes b')(c\otimes c'))$:

$$\begin{aligned}((a \otimes a')(b \otimes b'))(c \otimes c') &= (-1)^{|a'||b|}(ab \otimes a'b')(c \otimes c') \\ &= (-1)^{|a'||b|}(-1)^{|a'b'||c|}(ab)c \otimes (a'b')c' \\ &= (-1)^{|a'||b|+|a'||c|+|b'||c|}abc \otimes a'b'c' \\ (a \otimes a')((b \otimes b')(c \otimes c')) &= (-1)^{|b'||c|}(a \otimes a')(bc \otimes b'c') \\ &= (-1)^{|b'||c|}(-1)^{|a'|(|b|+|c|)}abc \otimes a'b'c' \\ &= (-1)^{|b'||c|+|a'||b|+|a'||c|}abc \otimes a'b'c'.\end{aligned}$$

Next we show that $A \otimes_R A'$ satisfies the Leibniz rule. This is equivalent to showing that μ is a chain map. Let $a \otimes a', b \otimes b' \in A \otimes_R A'$. We have the following computations:

$$\begin{aligned}&\partial^{A\otimes_R A'}((a \otimes a')(b \otimes b')) \\ &\qquad= \partial^{A\otimes_R A'}((-1)^{|a'||b|}ab \otimes a'b') \\ &\qquad= (-1)^{|a'||b|}\partial^{A\otimes_R A'}(ab \otimes a'b') \\ &\qquad= (-1)^{|a'||b|}\Big(\partial^{A}(ab) \otimes a'b' + (-1)^{|a|+|b|}ab \otimes \partial^{A'}(a'b')\Big) \\ &\qquad= (-1)^{|a'||b|}\Big(\partial^{A}(a)b \otimes a'b' + (-1)^{|a|}a\partial^{A}(b) \otimes a'b' \\ &\qquad\qquad + (-1)^{|a|+|b|}ab \otimes \partial^{A'}(a')b' + (-1)^{|a|+|b|+|a'|}ab \otimes a'\partial^{A'}(b')\Big)\end{aligned}$$

where the first step follows from multiplication of tensors. Because the elements a' and b switch order we must multiply by $(-1)^{|a'||b|}$. We pull this term out of the differential in the second step. In the third step we apply the differential to the tensor product; see Remark 11.2 (ii) for the map. For the fourth step, we apply the Leibniz rule using the fact that A and A' are DG algebras. We now compute the right hand side of equation eq11.4.1 to complete the proof:

$$\begin{aligned}&\partial^{A\otimes_R A'}(a \otimes a')(b \otimes b') + (-1)^{|a|+|a'|}(a \otimes a')\partial^{A\otimes_R A'}(b \otimes b') \\ &\qquad= \Big(\partial^{A}(a) \otimes a' + (-1)^{|a|}a \otimes \partial^{A'}(a')\Big)(b \otimes b') \\ &\qquad\qquad + (-1)^{|a|+|a'|}(a \otimes a')\Big(\partial^{A}(b) \otimes b' + (-1)^{|b|}b \otimes \partial^{A'}(b')\Big) \\ &\qquad= (-1)^{|a'||b|}\partial^{A}(a)b \otimes a'b' + (-1)^{|a|+|\partial^{A'}(a')||b|}ab \otimes \partial^{A'}(a')b' \\ &\qquad\qquad + (-1)^{|a|+|a'|+|a'||\partial^{A}(b)|}a\partial^{A}(b) \otimes a'b' \\ &\qquad\qquad + (-1)^{|a|+|a'|+|a'||b|+|b|}ab \otimes a'\partial^{A'}(b').\end{aligned}$$

The first step follows from applying the differential; see Remark 11.2(ii) for the map, and the second step follows from multiplication of tensors. Note that $|\partial^{A'}(a')| = |a'| - 1$ and $|\partial^{A}(b)| = |b| - 1$. By comparing the left and right hand side, we see that they are equal. Therefore $A \otimes_R A'$ is a DG algebra. □

Remark 12.2. We do have to question the well-definedness of the tensor product. The proof is short but not obvious. The following isomorphism is from the commutativity of tensor product

$$(A_i \otimes_R A'_j) \otimes_R (A_p \otimes_R A'_q) \cong (A_i \otimes_R A_p) \otimes_R (A'_j \otimes_R A'_q).$$

The definition of the tensor product is a combination of the above isomorphism with the next two maps

$$(A_i \otimes_R A_p) \otimes_R (A'_j \otimes_R A'_q) \to A_{i+p} \otimes_R A'_{j+q} \hookrightarrow (A \otimes_R A')_{i+j+p+q}.$$

We need to check that the sign changes of these are the same as in the product.

Remark 12.3. There exists a commutative diagram of DG algebra homomorphisms (base change)

$$\begin{array}{ccccc} R & \longrightarrow & A & & a \\ \downarrow & & \downarrow & & \downarrow \\ A' & \longrightarrow & A \otimes_R A' & & a \otimes 1 \end{array}$$

$$a' \longmapsto 1 \otimes a'.$$

This diagram shows that the algebra structure of the tensor product is very nice because of the commutativity of the tensor product. We have $r(1 \otimes 1) = r1 \otimes 1 = 1 \otimes r1$ for any $r \in R$.

Definition 12.4. A *DG module* M over a DG algebra A (otherwise written as a DG A-module) is an R-complex with a morphism $A \otimes_R M \to M$, defined by $a \otimes m \mapsto am$, that satisfies the following

(i) Leibniz rule: $\partial^M_{i+j}(am) = \partial^A_i(a)m + (-1)^i a \partial^M_j(m)$;
(ii) unitary: there exists $1 \in M$ such that $am = m = m1$ for all $m \in M$; and
(iii) associative: for all $a, b, c \in M$ we have $a(bc) = (ab)c$.

Example 12.5. An R-module is a DG R-module. A DG R-module is an R-complex. Apply the Leibniz rule to an element of R and we always get zero since R is in degree zero.

Remark 12.6. Let $f : M \to N$ be a homomorphism of R-complexes. Then we have $f \in \operatorname{Hom}_R(M, N)_i$ and f is of degree i. We write $|f| = i$.

Definition 12.7. An *A-homomorphism* of DG modules is a homomorphism of R-complexes $f : M \to N$ such that $f(am) = (-1)^{|f||a|} a f(m)$, that is, f is A-linear.

Remark 12.8. The homomorphism set of two DG A-algebras, $\operatorname{Hom}_A(M, N)$, is a DG A-module. We leave the proof of this as an exercise for the reader. The A-module structure is defined for $\phi \in \operatorname{Hom}_A(M, N)$ and $a \in A$ by $(a\phi)(m) = (-1)^{|\phi||a|} \phi(am) = a\phi(m)$. Note that $\operatorname{Hom}_A(M, N) \subseteq \operatorname{Hom}_R(M, N)$.

13. Day 13

Definition 13.1. Let A be a DG R-algebra and let M and N be DG A-modules. We define $M \otimes_A N := (M \otimes_R N)/L$ where L is generated by all elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$ where $a \in A$, $m \in M$, and $n \in N$. We check two things – that $M \otimes_A N$ respects the differential and that $M \otimes_A N$ is a DG A-module.

For the differentials, we have the chain complex

$$\cdots \to (M\otimes_A N)_i \xrightarrow{\partial_i^{M\otimes_A N}} (M\otimes_A N)_{i-1} \to \cdots .$$

Let $(am)\otimes n - (-1)^{|a||m|} m\otimes (an) \in (M\otimes_A N)_i$. We show that the differential maps this element into $(M\otimes_A N)_{i-1}$. We have the following computations

$$\begin{aligned}
&\partial^{M\otimes_A N}((am)\otimes n - (-1)^{|a||m|} m\otimes (an))\\
&\quad= \partial^{M\otimes_A N}(am\otimes n) - (-1)^{|a||m|}\partial^{M\otimes_A N}(m\otimes an)\\
&\quad= \partial^M(am)\otimes n + (-1)^{|a|+|m|} am\otimes \partial^N(n)\\
&\qquad - (-1)^{|a||m|}\big(\partial^M(m)\otimes an + (-1)^{|m|} m\otimes \partial^N(an)\big)\\
&\quad= \partial^A(a)m\otimes n + (-1)^{|a|} a\partial^M(m)\otimes n\\
&\qquad + (-1)^{|a|+|m|} am\otimes \partial^N(n) - (-1)^{|a||m|}\partial^M(m)\otimes an\\
&\qquad - (-1)^{|a||m|+|m|}(m\otimes \partial^A(a)n + (-1)^{|a|} m\otimes a\partial^N(n))\\
&\quad= \big(\partial^A(a)m\otimes n - (-1)^{|a||m|+|m|} m\otimes \partial^A(a)n\big)\\
&\qquad + \big((-1)^{|a|}(a\partial^M(m)\otimes n - (-1)^{|a||m|-|a|}\partial^M(m)\otimes an)\big)\\
&\qquad + \big((-1)^{|a|+|m|}(am\otimes \partial^N(n) - (-1)^{|a||m|} m\otimes a\partial^N(n))\big)
\end{aligned}$$

where the third and fourth computations follow from the Leibniz rule since M and N are both DG A-modules. The first and second steps are by definition. Note that for the signs we have the equalities $|a||m|+|m| = |a||m|-|m| = (|a|-1)|m| = |\partial^A(a)||m|$ and in a similar manner $|a||m| - |a| = |a||\partial^M(m)|$.

Next we check that $M\otimes_A N$ is a DG A-module. It suffices to show that $M\otimes_A N$ satisfies the Leibniz rule as the other two conditions are straightforward. Let $a\in A_i$, $m\in M_j$, and $n\in N_k$. We show the equality

$$\partial^{M\otimes_A N}(a(m\otimes n)) = \partial_i^A(a)(m\otimes n) + (-1)^i a\partial_{j+k}^{M\otimes_A N}(m\otimes n).$$

The third step in the following computations follows from the fact that M is a DG A-algebra:

$$\begin{aligned}
\text{LHS} &= \partial_{i+j+k}^{M\otimes_A N}((am)\otimes n)\\
&= \partial_{i+j}^M(am)\otimes n + (-1)^{i+j} am\otimes \partial_k^N(n)\\
&= \partial_i^A(a)m\otimes n + (-1)^i a\partial_j^M(m)\otimes n + (-1)^{i+j} am\otimes \partial_k^N(n)\\
\text{RHS} &= \partial_i^A(a)(m\otimes n) + (-1)^i a\partial_{j+k}^{M\otimes_A N}(m\otimes n)\\
&= \partial_i^A(a)m\otimes n + (-1)^i a(\partial_j^M(m)\otimes n + (-1)^j m\otimes \partial_k^N(n))\\
&= \partial_i^A(a)m\otimes n + (-1)^i a\partial_j^M(m)\otimes n + (-1)^{i+j} am\otimes \partial_k^N(n).
\end{aligned}$$

Therefore $M\otimes_A N$ is a DG A-algebra.

Next we tackle base change.

Proposition 13.2. *Let $A\to B$ be a morphism of DG R-algebras. Let M be a DG A-module. Then $B\otimes_A M$ has the structure of a DG B-module by the action $b(b'\otimes m) = (bb')\otimes m$.*

Proof. For well-definedness we need to show that if $b\sum_i(b_i'\otimes m_i) = b\sum_j(b_j''\otimes m_j')$ then we have $\sum_i(bb_i')\otimes m_i = \sum_j(bb_j'')\otimes m_j'$. This is shown in Lemma 14.3.

We next check the Leibniz rule. We show the equality

$$\partial_{i+j+k}^{B\otimes_A M}(b(b'\otimes m)) = \partial_i^B(b)(b'\otimes m) + (-1)^i b\partial_{j+k}^{B\otimes_A M}(b'\otimes m)$$

where b is in degree i, b' is in degree j, and m is in degree k. We have the following computations:

$$\begin{aligned}
\text{LHS} &= \partial_{i+j+k}^{B\otimes_A M}(b(b'\otimes m)) \\
&= \partial_{i+j}^B(bb')\otimes m + (-1)^{i+j}bb'\otimes \partial_k^M(m) \\
\text{RHS} &= \partial_i^B(b)b'\otimes m + (-1)^i b(\partial_j^B(b')\otimes m + (-1)^j b'\otimes \partial_k^M(m)) \\
&= \partial_i^B(b)b'\otimes m + (-1)^i b\partial_j^B(b')\otimes m + (-1)^{i+j}bb'\otimes \partial_k^M(m).
\end{aligned}$$

The left and right hand side are equal since the Leibniz rule with respect to B (since B is a DG R-module) implies the equality $\partial_{i+j}^B(bb')\otimes m = \partial_i^B(b)b'\otimes m + (-1)^i b\partial_j^B(b')\otimes m$. □

Corollary 13.3. *If M is an R-complex then $K(\underline{a})\otimes_R M$ is a DG $K(\underline{a})$-module.*

Proposition 13.4. *If M and N are DG A-modules with $A\to B$ and $f: M\to N$ is a homomorphism of DG A-modules, then the induced map*

$$B\otimes_A f : B\otimes_A M \to B\otimes_A N$$

is a DG B-morphism.

Proof. The well-definedness of this map is treated in Fact 14.2.

We show the equality $(B\otimes_A f)(a(b\otimes m)) = (-1)^{|B\otimes_A f||a|}a(B\otimes_A f)(b\otimes m)$ where $a,b\in B$ and $m\in M$. Note that degree wise we have $|B\otimes_A f| = |f|$ since $|B| = 1$ as the identity map. We have the following computations:

$$\begin{aligned}
\text{LHS} &= (B\otimes_A f)(ab\otimes m) \\
&= (-1)^{|f||a|+|f||b|}(ab\otimes f(m)) \\
\text{RHS} &= (-1)^{|f||b|}((-1)^{|f||a|}a(b\otimes f(m))) \\
&= (-1)^{|f||a|+|f||b|}(ab\otimes f(m)).
\end{aligned}$$

□

Corollary 13.5. *If M and N are R-complexes, and $f: M\to N$ is a chain map, then the induced map*

$$K(\underline{a})\otimes_R f : K(\underline{a})\otimes_R M \to K(\underline{a})\otimes_R N$$

is a DG $K(\underline{a})$-homomorphism.

14. Day 14

Fact 14.1. Let A be a DG R-algebra and let M and N be DG A-modules. Using $M\otimes_A N$ as defined in Definition 13.1, we check that the action $a(m\otimes n) := am\otimes n$ is well-defined. Note that we always have the equality $am\otimes n = (-1)^{|a||m|}m\otimes an$.

Suppose that $\sum m_i\otimes n_i = \sum m_i'\otimes n_i'$ for some $m_i\otimes n_i, m_i'\otimes n_i' \in M\otimes_A N$. Consider the multiplication map $\mu^a : M\to M$ defined by $m\mapsto am$ of $\deg \mu^a = |a|$ and the identity map $Id_N : N\to N$. By definition we have the equality

$$a(m\otimes n) = (\mu^a\otimes Id_N)(m\otimes n).$$

Until now all our tensors have been over A. Recall from Definition 13.1 that $M \otimes_A N = (M \otimes_R N)/L$. Since $\sum m_i \otimes n_i = \sum m_i' \otimes n_i'$ as a tensor over A, this implies that $\sum m_i \otimes n_i - \sum m_i' \otimes n_i' = 0$. In other words, $\sum m_i \otimes n_i - \sum m_i' \otimes n_i' \in L$ where the tensor is over R. Since L is generated by elements of a particular form, we can write $\sum m_i \otimes_R n_i - \sum m_i' \otimes_R n_i' = \sum a_i m_i'' \otimes_R n_i'' - (-1)^{|a_i||m_i''|} m_i'' \otimes_R a_i n_i''$ for some $a_i \in A$ and $m_i'' \otimes n_i'' \in M \otimes_R N$.

This implies that we have the following equalities in L

$$\begin{aligned}
(\mu^a \otimes Id_N)\big(\sum m_i \otimes_R n_i - \sum m_i' \otimes_R n_i'\big) &= \sum \big((\mu^a \otimes Id_N)(a_i m_i'' \otimes n_i'') \\
&\quad - (-1)^{|a_i||m_i''|}(\mu^a \otimes Id_N)(m_i'' \otimes a_i n_i'')\big) \\
&= \sum \big(\mu^a(a_i m_i'') \otimes n_i'' \\
&\quad - (-1)^{|a_i||m_i''|}\mu^a(m_i'') \otimes a_i n_i''\big) \\
&= \sum \big((-1)^{|a_i||\mu^a|} a_i \mu^a(m_i'') \otimes n_i'' \\
&\quad - (-1)^{|a_i||m_i''|}\mu^a(m_i'') \otimes a_i n_i''\big) \\
&= \sum (-1)^{|a_i||\mu^a|}\big(a_i \mu^a(m_i'') \otimes n_i'' \\
&\quad - (-1)^{|a_i||\mu^a(m_i'')|}\mu^a(m_i'') \otimes a_i n_i''\big) \\
&= 0.
\end{aligned}$$

This implies that $(\mu^a \otimes Id_N)(\sum m_i \otimes n_i) = (\mu^a \otimes Id_N)(\sum m_i' \otimes n_i')$ and hence we conclude that $a \sum m_i \otimes n_i = a \sum m_i' \otimes n_i'$.

Fact 14.2. Let $f : M \to M'$ and $g : N \to N'$ be DG A-homomorphisms of DG A-modules of degrees t and s respectively. Then $f \otimes g : M \otimes_A N \to M' \otimes_A N'$ given by $(f \otimes g)(m \otimes n) = (-1)^{|m||g|} f(m) \otimes g(n)$ is a well-defined DG A-homomorphism.

Applying Definition 13.1 we have $M \otimes_A N = (M \otimes_R N)/L$ and $M' \otimes_A N' = (M' \otimes_R N')/L'$. First we prove that $f \otimes g$ is well-defined. Suppose that $\sum m_i \otimes n_i = \sum p_i \otimes q_i$ for some $m_i \otimes n_i, p_i \otimes q_i \in M \otimes_A N$. Then we have $\sum m_i \otimes n_i - \sum p_i \otimes q_i = \sum (a_i m_i'' \otimes n_i'' - (-1)^{|a_i||m_i''|} m_i'' \otimes a_i n_i'')$ for some $a_i \in A$ and $m_i'' \otimes n_i'' \in M \otimes_R N$.

This implies we have the following equalities in L':

$$
\begin{aligned}
&(f\otimes g)\big(\sum m_i\otimes n_i-\sum p_i\otimes q_i\big)\\
&\qquad=(f\otimes g)\big(\sum a_im_i''\otimes n_i''-(-1)^{|a_i||m_i''|}m_i''\otimes a_in_i''\big)\\
&\qquad=\sum\big((f\otimes g)(a_im_i''\otimes n_i'')-(-1)^{|a_i||m_i''|}(f\otimes g)(m_i''\otimes a_in_i'')\big)\\
&\qquad=\sum\big((-1)^{|g||a_i|+|g||m_i''|}f(a_im_i'')\otimes g(n_i'')\\
&\qquad\qquad-(-1)^{|a_i||m_i''|+|g||m_i''|}f(m_i'')\otimes g(a_in_i'')\big)\\
&\qquad=\sum\big((-1)^{|g||a_i|+|g||m_i''|+|a_i||f|}a_if(m_i'')\otimes g(n_i'')\\
&\qquad\qquad-(-1)^{|a_i||m_i''|+|g||m_i''|+|a_i||g|}f(m_i'')\otimes a_ig(n_i'')\big)\\
&\qquad=\sum(-1)^{|g||a_i|+|g||m_i''|+|a_i||f|}\big(a_if(m_i'')\otimes g(n_i'')\\
&\qquad\qquad-(1)^{|a_i||m_i''|-|a_i||f|}f(m_i'')\otimes a_ig(n_i'')\big)\\
&\qquad=\sum(-1)^{|g||a_i|+|g||m_i''|+|a_i||f|}\big(a_if(m_i'')\otimes g(n_i'')\\
&\qquad\qquad-(1)^{|a_i||m_i''|+|a_i||f|}f(m_i'')\otimes a_ig(n_i'')\big)\\
&\qquad=\sum(-1)^{|g||a_i|+|g||m_i'''|+|a_i||f|}\big(a_if(m_i'')\otimes g(n_i'')\\
&\qquad\qquad-(-1)^{|a_i||f(m_i'')|}f(m_i'')\otimes a_ig(n_i'')\big).
\end{aligned}
$$

Thus we conclude that $(f\otimes g)(\sum m_i\otimes n_i)=(f\otimes g)(\sum p_i\otimes q_i)$ and hence $f\otimes g$ is well-defined.

Next we prove that $f\otimes g$ is an A-homomorphism. We show $(f\otimes g)(a(m\otimes m'))=(-1)^{|f\otimes g||a|}a(f\otimes g)(m\otimes m')$. For the left hand side of the equality we have the following equalities:

$$
\begin{aligned}
(f\otimes g)(am\otimes m')&=(-1)^{|g||am|}f(am)\otimes g(m')\\
&=(-1)^{|g|(|a|+|m|)}f(am)\otimes g(m')\\
&=(-1)^{|g|(|a|+|m|)+|f||a|}af(m)\otimes g(m').
\end{aligned}
$$

The last equality follows from the fact that f is a DG A-homomorphism. For the right hand side we have the following:

$$
\begin{aligned}
(-1)^{|f\otimes g||a|}a(f\otimes g)(m\otimes m')&=(-1)^{(|f|+|g|)|a|}a(-1)^{|g||m|}f(m)\otimes g(m')\\
&=(-1)^{|f||a|+|g||a|+|g||m|}af(m)\otimes g(m').
\end{aligned}
$$

We conclude that $f\otimes g$ is an A-homomorphism.

Now we return to the previous result of base change from Proposition 13.2 to show that the action described is well defined.

Lemma 14.3. *Let $A\to B$ be a morphism of DG R-algebras. Let M be a DG A-module. Then the DG B-module $B\otimes_A M$ defined by the action $b(b'\otimes m)=(bb')\otimes m$ is well-defined.*

Proof. Consider the DG A-homomorphisms $Id_M:M\to M$ and $\mu^b:B\to B$ of degrees 0 and $|b|$ respectively where μ^b is defined by the map $b'\mapsto bb'$ for all $b'\in B$. Fact 14.2 implies that the map $\mu^b\otimes Id_M:B\otimes_A M\to B\otimes_A M$ is a DG A-module homomorphism. We have the equality

$$(\mu^b \otimes Id_M)(b' \otimes m) = (bb') \otimes m$$

so in fact $b(b' \otimes m) = (\mu^b \otimes Id_M)(b' \otimes m)$ which implies that the action is well-defined. $\square$

Proposition 14.4. *If M and N are DG A-modules and $f : M \to N$ is a DG A-homomorphism and also $A \to B$ is a homomorphism between DG algebras, then the induced map*

$$B \otimes_A f : B \otimes_A M \to B \otimes_A N$$

is a DG B-homomorphism.

Proof. Proposition 13.2 implies that $B \otimes_A M$ and $B \otimes_A N$ are DG B-modules so we only need to check $(B \otimes_A f)(a(b \otimes m)) = (-1)^{|B\otimes f||a|}a(B \otimes_A f)(b \otimes m)$ for all $a, b \in B$ and $m \in M$. We have the following computations

$$\begin{aligned}(B \otimes_A f)(ab \otimes m) &= (-1)^{|f|(|a|+|b|)} ab \otimes f(m) \\ &= (-1)^{|f||a|} a(-1)^{|f||b|} b \otimes f(m) \\ &= (-1)^{|B\otimes f||a|} a(B \otimes_A f)(b \otimes m)\end{aligned}$$

where the first step follows from Fact 14.2 and the second step follows from Proposition 13.2. The third step follows from the fact that $|f| = |B \otimes f|$.

$\square$

The next definition is equivalent to to one given in [1, pages 29-30].

Definition 14.5. Let R be a commutative ring with identity. An *R-algebra* is an abelian group S which admits mapping $S \times S \to S$ defined by $(a, b) \mapsto ab$ and a mapping $R \times S \to S$ defined by $(r, a) \mapsto ra$ that satisfy the following conditions:

(1) $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a, b, c \in S$.
(2) $r(a + b) = ra + rb$ for all $r \in R$ and $a, b \in S$.
(3) $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in S$.
(4) $(rs)a = r(sa)$ for all $r, s \in R$ and $a \in S$.
(5) $1a = a$ for all $a \in S$.

Remark 14.6. Note that A_0 is an R-algebra. The natural map $A_0 \to A$ is a morphism of DG algebras as described below:

$$\begin{array}{ccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & A_0 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle Id} & & \\ \cdots & \xrightarrow{\partial_2^A} & A_1 & \xrightarrow{\partial_1^A} & A_0 & \xrightarrow{\partial_0^A} & 0. \end{array}$$

The condition $A_{-1} = 0$ implies that $\operatorname{Ker}(\partial_0^A) = A_0$. Hence A_0 surjects onto $H_0(A)$ and $H_0(A)$ is an A_0-algebra.

Furthermore, the R-module A_i is an A_0-module and $H_i(A)$ is an $H_0(A)$-module for each i by the following action: let $a + \operatorname{Im}(\partial_1^A) \in \mathrm{H}_0(A) = A_0/\operatorname{Im}(\partial_1^A)$ and $b + \operatorname{Im}(\partial_{i+1}^A) \in \mathrm{H}_i(A) = \operatorname{Ker}(\partial_i^A)/\operatorname{Im}(\partial_{i+1}^A)$. Then we define

$$(a + \operatorname{Im}(\partial_1^A))(b + \operatorname{Im}(\partial_{i+1}^A)) := ab + \operatorname{Im}(\partial_{i+1}^A).$$

This is well-defined since $ab \in \operatorname{Ker}(\partial_i^A)$ as shown below. The first equality of the following sequence follows from the Leibniz rule:

$$\partial_i^A(ab) = \partial_0^A(a)b + (-1)^{|a|}a\partial_i^A(b) = a\partial_i^A(b) = 0.$$

The second equality follows from the fact that $a \in \operatorname{Ker}(\partial_0^A)$ and $|a| = 0$, and the third equality follows from the fact that $b \in \operatorname{Ker}(\partial_i^A)$.

Definition 14.7. Let A be a DG algebra. We say A is *noetherian* if $H_0(A)$ is noetherian and $H_i(A)$ is finitely generated as an $H_0(A)$-module for all $i \geqslant 0$. We say that $H_0(A)$ is a *local R-algebra* if $H_0(A)$ is a local ring whose maximal ideal contains the ideal $\mathfrak{m}H_0(A)$. We say A is *local* if it is noetherian and the ring $H_0(A)$ is a local R-algebra.

Proposition 14.8. *Let A be a local DG algebra. The composition*

$$A \to H_0(A) \to k$$

where $k = A_0/\mathfrak{m}_0$ for some $\mathfrak{m}_0 \subsetneq A_0$ has kernel of the form

$$\mathfrak{m}_A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{m}_0 \to 0.$$

Since $H_0(A)$ is a local R-algebra, we have $\mathfrak{m}A_0 \subset \mathfrak{m}_0$.

Proof. From the composition map $A \to H_0(A) \to k$, we have the following diagram:

$$\begin{array}{ccccccc}
\cdots \longrightarrow & A_1 & \xrightarrow{\partial_1^A} & A_0 & \longrightarrow & 0 \\
& \downarrow & & \twoheaddownarrow & & \\
\cdots \longrightarrow & 0 & \longrightarrow & A_0/\operatorname{Im}(\partial_0^A) & \longrightarrow & 0 \\
& \downarrow & & \downarrow & & \\
\cdots \longrightarrow & 0 & \longrightarrow & A_0/\mathfrak{m}_0 & \longrightarrow & 0.
\end{array}$$

Since $\operatorname{Im}(\partial_1^A) \subseteq A_0$, there exists $\mathfrak{m}_0 \in \max(A_0)$ such that $\operatorname{Im}(\partial_1^A) \subseteq \mathfrak{m}_0$. Set $k = A_0/\mathfrak{m}_0$. The map $A_0/\operatorname{Im}(\partial_1^A) \to A_0/\mathfrak{m}_0$ defined by $a + \operatorname{Im}(\partial_A^1) \mapsto a + \mathfrak{m}_0$ for any $a \in A_0$ is well-defined. It is clear that

$$\operatorname{Ker}(A \to H_0(A) \to k) = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} \mathfrak{m}_0 \to 0.$$

Since $\mathrm{H}_0(A)$ is a local R-algebra, $\mathfrak{m}\,\mathrm{H}_0(A) = \mathfrak{m}(A_0/\operatorname{Im}(\partial_1^A)) \subseteq \mathfrak{m}_0/\operatorname{Im}(\partial_1^A)$. Because $\operatorname{Im}(\partial_1^A) \subseteq \mathfrak{m}_0$, we have that $\mathfrak{m}A_0 \subseteq \mathfrak{m}_0$. □

Definition 14.9. Given a local DG R-algebra A, the sub-complex $\mathfrak{m}_A$ is called the *augmentation ideal* of A.

15. Day 15

Proposition 15.1. *Let $A \to B$ be a DG homomorphism of DG R-algebras and let M and N be DG A-modules. Then the map*

$$B \otimes_A \operatorname{Hom}_A(M, N) \xrightarrow{\xi} \operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)$$

defined by $\xi(b \otimes f)(b' \otimes m) = (-1)^{|b'||f|}bb' \otimes f(m)$ is a DG homomorphism.

Proof. We need to show the map is well-defined, ξ is a DG homomorphism, and ξ is a chain homomorphism.

First we check well-definedness. Let $\sum b_i \otimes f_i = \sum c_i \otimes g_i$ in $B \otimes_A \operatorname{Hom}_A(M,N)$. Then we have $\sum b_i \otimes f_i - \sum c_i \otimes g_i = \sum \big(a_i d_i \otimes h_i - (-1)^{|a_i||d_i|} d_i \otimes a_i h_i\big)$ for some $a_i \in A$ and $d_i \otimes h_i \in B \otimes_A \operatorname{Hom}_A(M,N)$ by Definition 13.1. Then we have the following computations:

$$\begin{aligned}
&\xi\Big(\sum\big(a_i d_i \otimes h_i - (-1)^{|a_i||d_i|} d_i \otimes a_i h_i\big)\Big)(b' \otimes m') \\
&= \sum\big(\xi(a_i d_i \otimes h_i)(b' \otimes m') - (-1)^{|a_i||d_i|}\xi(d_i \otimes a_i h_i)(b' \otimes m')\big) \\
&= \sum\big((-1)^{|h_i||b'|} a_i d_i b' \otimes h_i(m') - (-1)^{|a_i||d_i|+|a_i||b'|+|h_i||b'|} d_i b' \otimes (a_i h_i)(m')\big) \\
&= \sum (-1)^{|h_i||b'|}\big(a_i(d_i b') \otimes h_i(m') - (-1)^{|d_i b'||a_i|} d_i b' \otimes a_i h_i(m')\big) \\
&= 0.
\end{aligned}$$

Therefore we conclude that $\xi(\sum b_i \otimes f_i) = \xi(\sum c_i \otimes g_i)$ and the map is well-defined. Although we do not show that the map is independent of $b' \otimes m'$, this follows from last time.

Next we show that ξ is a DG homomorphism. The proof to show that ξ is an DG A-homomorphism is similar to the proof to show that ξ is a DG B-homomorphism. We show that ξ is a DG A-homomorphism and leave the other case to the reader.

We show that $\xi(a(b \otimes f)) = (-1)^{|\xi||a|} a\xi(b \otimes f)$. Since $\deg \xi = 0$, we have $(-1)^{|\xi||a|} = 1$. We have the following computations

$$\begin{aligned}
\xi(a(b \otimes f))(b' \otimes m') &= \xi(ab \otimes f)(b' \otimes m') \\
&= (-1)^{|f||b'|} abb' \otimes f(m') \\
&= a(-1)^{|f||b'|} bb' \otimes f(m') \\
&= a(\xi(b \otimes f)(b' \otimes m')) \\
&= (a\xi(b \otimes f))(b' \otimes m') \\
&= (-1)^{|\xi||a|} a\xi(b \otimes f)(b' \otimes m').
\end{aligned}$$

Therefore ξ is a DG A-homomorphism. Note that to be a DG B-homomorphism is stronger than to be a DG A-homomorphism.

Next we show that ξ is a chain homomorphism. Let $\lambda_A^M : A \to \operatorname{Hom}_A(M,M)$ be defined by $\lambda_A^M(a)(m) = am$. We show that this is a homomorphism of DG A-modules. Keep in mind that $|\lambda_A^M| = 0$ and $a, b \in A$ are of different degrees. We have $\lambda_A^M(ab)(m) = abm = a\lambda_A^M(b)(m)$ and we do not need parenthesis in the middle step because of the associative property. We have the following commutative diagram:

$$\begin{array}{ccc}
(B \otimes_A \operatorname{Hom}_A(M,N))_i & \xrightarrow{\xi_i} & (\operatorname{Hom}_B(B \otimes_A M, B \otimes_A N))_i \\
\Big\downarrow \partial_i^{B \otimes_A \operatorname{Hom}_A(M,N)} & & \Big\downarrow \partial_i^{\operatorname{Hom}_B(B \otimes_A M, B \otimes_A N)} \\
(B \otimes_A \operatorname{Hom}_A(M,N))_{i-1} & \xrightarrow{\xi_{i-1}} & (\operatorname{Hom}_B(B \otimes_A M, B \otimes_A N))_{i-1}.
\end{array}$$

It is enough to consider a generator $b \otimes f \in (B \otimes_A \operatorname{Hom}_A(M,N))_i$. We have the following computations:

$$
\begin{aligned}
&(\xi_{i-1}\partial_i^{B\otimes_A \operatorname{Hom}_A(M,N)}(b\otimes f))(b'\otimes m)\\
&\quad = \xi_{i-1}(\partial_i^B(b)\otimes f + (-1)^{|b|} b\otimes \partial_i^{\operatorname{Hom}_A(M,N)}(f))(b'\otimes m)\\
&\quad = (-1)^{|f||b'|}\partial_i^B(b)b'\otimes f(m) + (-1)^{|b|+(|f|-1)|b'|} bb'\otimes \partial_i^{\operatorname{Hom}_A(M,N)}(f)(m).
\end{aligned}
$$

For the other direction around the square we begin with

$$\partial_i^{\operatorname{Hom}_B(B\otimes_A M, B\otimes_A N)}(\xi_i(b\otimes f))(b'\otimes m).$$

The details are left to the reader. □

16. Day 16

Example 16.1. Consider the Koszul complex $K(x,y,z)$. We have the following diagram:

$$0 \to R \xrightarrow{\partial_3} R^3 \xrightarrow{\partial_2} R^3 \xrightarrow{\partial_1} R \to 0.$$

In degree three we have the element $e_1 \wedge e_2 \wedge e_3$, in degree two we have the elements $e_1 \wedge e_2$, $e_1 \wedge e_3$, and $e_2 \wedge e_3$, in degree one we have the elements e_1, e_2, and e_3, and in degree zero we have the element 1.

We have the following computation:

$$\partial_3(e_1\wedge e_2\wedge e_3) = xe_2\wedge e_3 - ye_1\wedge e_3 + ze_1\wedge e_2.$$

This implies that $\partial_3 = \begin{pmatrix} z \\ -y \\ x \end{pmatrix}$. In a similar manner we find that $\partial_1 = \begin{pmatrix} x & y & z \end{pmatrix}$. We have the following computations:

$$
\begin{aligned}
\partial_3(e_1\wedge e_2\wedge e_3) &= \partial_3(e_1(e_2\wedge e_3))\\
&= \partial_1(e_1)e_2\wedge e_3 + (-1)^{|e_1|}e_1\partial_2(e_2\wedge e_3)\\
&= xe_2\wedge e_3 - (e_1(ye_3 - ze_2))\\
&= xe_2\wedge e_3 - ye_1\wedge e_3 + ze_1\wedge e_2
\end{aligned}
$$

We would like to do homological algebra in this setting with Ext_A and Tor^A. If M is a DG A-module, we can use Ext_A and Tor^A since we know that a resolution of M is an R-complex. Our next step is to understand the resolution of a complex in order to use Ext_A and Tor^A with respect to complexes.

Remark 16.2. Let T be an R-module. Then a free resolution of T will look like the following:

$$F = \cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \to 0$$

where F_i is a free modules for each i. The augmented resolution is as follows:

$$F^+ = \cdots \xrightarrow{\partial_2^F} F_1 \xrightarrow{\partial_1^F} F_0 \xrightarrow{\tau} T \to 0.$$

Then the map $F \xrightarrow{\simeq} T$ is given by

$$\begin{array}{ccccccc} \cdots & \xrightarrow{\partial_2^F} & F_1 & \xrightarrow{\partial_1^F} & F_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle \tau} & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & T & \longrightarrow & 0. \end{array}$$

This is a good perspective from which to define a free resolution of an R-complex.

Definition 16.3. If U is an R-complex, then a *free resolution of U over R* is a quasiisomorphism $G \xrightarrow{\simeq} U$ where G is a complex of free R-modules such that $G_i = 0$ for all $i \ll 0$.

Fact 16.4. The R-complex U has a free resolution if and only if $\mathrm{H}_i(U) = 0$ for all $i \ll 0$.

17. Day (April 13th)

Definition 17.1. Let $A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \to 0$ be a commutative DG R-algebra where R is a commutative noetherian ring. Let $M = \cdots \xrightarrow{\partial_{i+2}^M} M_{i+1} \xrightarrow{\partial_{i+1}^M} M_i \xrightarrow{\partial_i^M} \cdots$ be a DG A-module.

Then $A^\natural = \cdots \xrightarrow{0} A_2 \xrightarrow{0} A_1 \xrightarrow{0} A_0 \to 0$ is a DG R-algebra (strip away the differential) and $M^\natural = \cdots \xrightarrow{0} M_{i+1} \xrightarrow{0} M_i \xrightarrow{0} M_{i-1} \xrightarrow{0} \cdots$, called M natural, is a DG $A^\natural$-module.

A *basis* for $M^\natural$ is a subset $E \subseteq M^\natural$ (or a set of subsets $E_i \subseteq M_i$ where $i \in \mathbb{Z}$) such that every element $m_i \in M_i$ has a unique decomposition as a linear combination over $A^\natural$ in E. For all $m_i \in M_i$ there exists a unique $m_i = \sum_{e \in E}^{finite} a_e e$ such that $a_e \in A^\natural$.

A *semibasis* for M is a set $E \subseteq M$ such that E is a basis for $M^\natural$ over $A^\natural$. We say M is *semifree* over A if it is bounded below and has a semibasis.

Example 17.2. A semifree DG R-module is a bounded below complex of free R-modules.

Example 17.3. If the semibasis for M has two elements e_{01} and e_{02} in degree 0 and three elements e_{11}, e_{12}, and e_{13} in degree 1 then we have the following:

$$\begin{array}{cccccccccccccc} \text{degree} & \cdots & 4 & & 3 & & 2 & & 1 & & 0 & & \\ & & & & & & & & & & & & \\ M: & \cdots & A_4^2 \oplus A_3^3 & \longrightarrow & A_3^2 \oplus A_2^3 & \longrightarrow & A_2^2 \oplus A_1^3 & \longrightarrow & A_1^2 \oplus A_0^3 & \longrightarrow & A_0^2 & \longrightarrow & 0. \end{array}$$

The maps between the modules are homomorphisms; it is difficult to say more unless we are working with a specific A. For example, we have

$$A_3^2 \oplus A_2^3 \xrightarrow{\begin{pmatrix} \alpha_3 & \gamma_2 \\ \beta_3 & \delta_2 \end{pmatrix}} A_2^2 \oplus A_1^3$$

where

$$\alpha_3 : A_3^2 \to A_2^2$$
$$\beta_3 : A_3^2 \to A_1^3$$
$$\gamma_2 : A_2^3 \to A_2^2$$
$$\delta_2 : A_2^3 \to A_1^3.$$

Example 17.4. Let $K(x)$ be a Koszul complex and

$$\begin{array}{ccccccc} A = K(x) = 0 & \longrightarrow & K_1 & \longrightarrow & K_0 & \longrightarrow & 0 \\ & & \downarrow \cong & & \downarrow \cong & & \\ & & R & & R & & \end{array}$$

with basis elements $e \in K_1$ and $1 \in K_0$, and where $\partial_1^K(e) = x$. Let e_{01} and e_{02} in degree zero and e_{11}, e_{12}, and e_{13} in degree one be a semibasis for M. Then we have the following:

$$M = \quad 0 \longrightarrow \begin{matrix} 0 \\ \oplus \\ K_1^3 \end{matrix} \xrightarrow{\begin{pmatrix} \alpha \\ \beta \end{pmatrix}} \begin{matrix} K_1^2 \\ \oplus \\ K_0^3 \end{matrix} \xrightarrow{\begin{pmatrix} \gamma & \delta \end{pmatrix}} \begin{matrix} K_0^2 \\ \oplus \\ 0 \end{matrix} \longrightarrow 0$$

with basis elements

$$ee_{11}, ee_{12}, ee_{13} \in K_1^3$$
$$ee_{01}, ee_{02} \in K_1^2$$
$$e_{11}, e_{12}, e_{13} \in K_0^3$$
$$e_{01}, e_{02} \in K_0^2.$$

The fact that M must satisfy the Leibniz rule for differentials explains the second step in the following display:

$$e\alpha(e_{1j}) + \beta(e_{1j}) = \partial_2^M(ee_{1j}) = \partial_1^K(e)e_{1j} + (-1)^{|e|}e\partial_1^M(e_{1j}) = xe_{1j} - e\delta(e_{1j}).$$

The third step follows from the fact that $|e| = 1$ and the equality $\partial_1^M(e_{1j}) = \delta(e_{1j})$. Hence we have that

$$\begin{pmatrix} \alpha(e_{1j}) \\ \beta(e_{1j}) \end{pmatrix} = \begin{pmatrix} -\delta(e_{1j}) \\ xe_{1j} \end{pmatrix}$$

implying that $\alpha = -\delta$ and β is multiplication by x. For the Leibniz rule in degree one we have the following:

$$\gamma(e_{0j}) = \partial_1^M(ee_{0j}) = \partial_1^K(e)e_{0j} + (-1)^{|e|}e\partial_0^M(e_{0j}) = xe_{0j}.$$

We conclude that $\gamma = x$, that is, the map γ is defined by multiplication with the element x.

Proposition 17.5. *If M is a semifree DG R-module then $A \otimes_R M$ is a semifree DG A-module.*

Proof. Recall that $(A \otimes_R M)_i = \oplus_{p \in \mathbb{Z}}(A_p \otimes_R M_{i-p})$ with a DG A-module structure such that

$$a_j(a_p \otimes m_{i-p}) := (a_j a_p) \otimes m_{i-p}$$

where the element a_j is in degree j, the element $a_p \otimes m_{i-p}$ is in degree i, the element $a_j a_p$ is in degree $j+p$, and the element m_{i-p} is in degree $i-p$.

For the semibasis, let $e_q \in E_q \subseteq M_q$. Then $1 \otimes e_q \in A_0 \otimes_R M_q \subseteq (A \otimes_R M)_q$. Let $\tilde{E} = \{1 \otimes e_q | e_q \in E_q, q \in \mathbb{Z}\}$. Then $\tilde{E}$ forms a semibasis.

Let $\eta \in (A \otimes_R M)_i$. Then $\eta = \{\eta_p\}_{p\in\mathbb{Z}}$ where $\eta_p \in A_p \otimes_R M_{i-p}$ for all $p \in \mathbb{Z}$. We need to show that $\eta_p \in A_p \otimes_R M_{i-p}$ can be written uniquely as a linear combination of elements of the form $1 \otimes e_q$ with coefficients in A.

By assumption, M_{i-p} is a free R-module implying that we have $M_{i-p} \cong R^{(\Lambda_{i-p})}$ for some set Λ_{i-p}. Then $A_p \otimes_R M_{i-p} \cong A_p \otimes_R R^{(\Lambda_{i-p})} \cong (A_p)^{(\Lambda_{i-p})}$ where elements of the form $a_p \otimes e_{i-p} \in A_p \otimes_R R^{(\Lambda_{i-p})}$ can be written as $a_p e_{i-p} \in (A_p)^{(\Lambda_{i-p})}$. In particular, $a_p \otimes e_{i-p} = a_p(1 \otimes e_{i-p}) \in \tilde{E}$. Therefore the elements in $(A_p)^{(\Lambda_{i-p})}$ have a unique expression. Thus η_p has a unique expression as a finite sum of things of the form $a_p e_{i-p} \in (A_p)^{(\Lambda_{i-p})}$, that is, it has a unique expression as a linear combination with coefficients in A. (This is because $A_i = 0$ for all $i < 0$ and $M_i = 0$ for $i \ll 0$ hence $(A \otimes_R M)_i = 0$ for all $i \ll 0$.) □

Definition 17.6. If M is a DG A-module, a *semifree resolution* of M over A is a quasiisomorphism (in the category of DG A-module homomorphisms) $F \xrightarrow{\simeq} M$ such that F is a semifree DG A-module.

Fact 17.7. If M has a semifree resolution, then $\mathrm{H}_i(M) = 0$ for all $i \ll 0$ because $F_i = 0$ for all $i \ll 0$. The converse holds but is difficult to prove.

Lemma 17.8. *Let F be a bounded below complex of flat R-modules. Then the following hold:*

(a) *If X is an exact complex of R-modules, then $F \otimes_R X$ is exact.*
(b) *If $\phi : Y \to Z$ is a quasiisomorphism over R, then $F \otimes_R \phi : F \otimes_R Y \to F \otimes_R Z$ is a quasiisomorphism.*

Proof. We will only sketch the proof here and leave the details to the reader. For part (a), reduce to cases.

Case 1. F is bounded. In this case, there are integers a and b such that $a \leqslant b$ and $F_i = 0$ for all $i < a$ and all $i > b$. In this case, we argue by induction on $n = b - a$. For convenience, we say that F is "concentrated in a degree-interval of length $\leqslant n$".

Base case: $n = 0$. In this case, F is a flat module F_a concentrated in degree a, so the desired conclusion holds by the flatness of F_a.

Induction step. Assume that $n \geqslant 1$ and that the result holds for bounded complexes of flat modules that concentrated in a degree-interval of length $\leqslant n-1$. By assumption, we have

$$F = \quad 0 \to F_b \xrightarrow{\partial_b^F} \cdots \xrightarrow{\partial_{a+1}^F} F_a \to 0.$$

Consider the following complex obtained by "truncating" F:

$$F = \quad 0 \to F_b \xrightarrow{\partial_b^F} \cdots \xrightarrow{\partial_{a+2}^F} F_{a+1} \to 0.$$

This complex fits into a short exact sequence of chain maps

$$0 \to \Sigma^a F_a \to F \to F' \to 0. \tag{17.8.1}$$

Here $\Sigma^a F_a$ is the module F_a concentrated in degree a. The base case implies that $\Sigma^a F_a \otimes_R X$ is exact, and the induction hypothesis implies that $F' \otimes_R X$ is exact.

Use the long exact sequence associated to the sequence (17.8.1) to conclude that $F \otimes_R X$ is exact.

Case 2. The general case. Since F is bounded below, there is an integer c such that $F_i = 0$ for all $i < c$. We need to show that $\mathrm{H}_i(F \otimes_R X) = 0$ for all i. Let $i \in \mathbb{Z}$ be given. Show that there is an integer j (depending on c and i) such that the complex

$$F'' = \quad 0 \to F_j \xrightarrow{\partial_j^F} \cdots \xrightarrow{\partial_{c+1}^F} F_a \to 0$$

satisfies the following conditions: $(F'' \otimes_R X)_m = (F \otimes_R X)_m$ for $m = i-1, i, i+1$ and $\partial_m^{F'' \otimes_R X} = \partial_m^{F \otimes_R X}$ for $m = i, i+1$. (In other words, the complexes $F'' \otimes_R X$ and $F \otimes_R X$ are the same in degrees $i-1$, i, and $i+1$.) This explains the first step in the next display

$$\mathrm{H}_i(F \otimes_R X) \cong \mathrm{H}_i(F'' \otimes_R X) = 0$$

while the second step follows from Case 1 since F'' is a bounded complex of flat R-modules.

For part (b), apply part (a) to the mapping cone. □

Proposition 17.9. *If A_i is flat over R for all i and $F \xrightarrow{\simeq} M$ is a semifree resolution over R, then $A \otimes_R F \xrightarrow{\simeq} A \otimes_R M$ is a semifree resolution over A.*

Proof. We will only sketch the proof here and leave the details to the reader. Lemma 17.8(b) implies that the quasiisomorphism is respected. By Proposition 13.4 DG R-module homomorphisms tensor up to DG A-module homomorphisms. Proposition 17.5 implies that $A \otimes_R F$ is semifree over A. □

Corollary 17.10. *If $\underline{x} = x_1, \cdots, x_n \in R$ and $F \xrightarrow{\simeq} M$ is a semifree DG R-module resolution, then $K \otimes_R F \xrightarrow{\simeq} K \otimes_R M$ is a semifree DG K-module resolution where $K = K(\underline{x})$.*

Proof. This follows from the fact that $K_i \cong R^{\binom{n}{i}}$ is flat over R. □

18. Day (April 20th)

Definition 18.1. Let A be a DG R-algebra and let M be a DG A-module. Let $n \in \mathbb{Z}$. Then we define the shift of a DG A-module M by $(\Sigma^n M)_i = M_{i-n}$ and the differential $\partial_i^{\Sigma^n M} = (-1)^n \partial_{i-n}^M$. If $a_i \in A_i$ and $m \in (\Sigma^n M)_p = M_{p-n}$, then we define the star operation to be

$$a_i * m := (-1)^{in} a_i m \in (\Sigma^n M)_{i+p} = M_{i+p-n}.$$

Lemma 18.2. *Let A be a DG R-algebra and let M be a DG A-module. Then $\Sigma^n M$ is a DG A-module.*

Proof. We show that the Leibniz rule is satisfied and leave the other details to the reader.

Let $a_i \in A_i$ and $m \in (\Sigma^n M)_p$. Then we have the following:

$$\begin{aligned}
\partial_{i+p}^{\Sigma^n M}(a_i * m) &= \partial_{i+p}^{\Sigma^n M}((-1)^{in} a_i m) \\
&= (-1)^n \partial_{i+p-n}^M((-1)^{in} a_i m) \\
&= (-1)^{n+in}\big(\partial_i^A(a_i)m + (-1)^i a_i \partial_{p-n}^M(m)\big) \\
\partial_i^A(a_i) * m + (-1)^i a_i * \partial_p^{\Sigma^n M}(m) &= (-1)^{(i-1)n}\partial_i^A(a_i)m + (-1)^{i+n} a_i * \partial_{p-n}^M(m) \\
&= (-1)^{in-n}\partial_i^A(a_i)m + (-1)^{i+n}(-1)^{in} a_i \partial_{p-n}^M(m).
\end{aligned}$$

Since $in - n \equiv in + n \pmod 2$ and $n + in + i \equiv i + n + in \pmod 2$, we conclude the the Leibniz rule is satisfied. □

Definition 18.3. Let A be a DG R-algebra and let M and N be DG A-modules. Let $\alpha : M \to N$ be a morphism of DG A-modules. Then $\Sigma^n \alpha : \Sigma^n M \to \Sigma^n N$ is defined by $(\Sigma^n \alpha)_p(m) := \alpha_{p-n}(m)$. Note that there is no sign change since the degree of α is zero (implying that the sign is trivial).

The proof of the next lemma is left as an exercise for the reader.

Lemma 18.4. *$\Sigma^n \alpha$ is a morphism, that is, it is A-linear.*

Definition 18.5. Let A be a DG R-algebra and let M and N be DG A-modules. Let $\alpha : M \to N$ be a morphism of DG A-modules. Then we define

$$\operatorname{Cone}(\alpha)_i = \begin{matrix} N_i \\ \oplus \\ M_{i-1} \end{matrix} \qquad \text{and} \qquad \partial_i^{\operatorname{Cone}(\alpha)} = \begin{pmatrix} \partial_i^N & \alpha_{i-1} \\ 0 & -\partial_{i-1}^M \end{pmatrix}.$$

Fix $a_i \in A_i$. Let $n_j \in N_j$ and let $m_{j-1} \in M_{j-1}$. Then we define

$$a_i \begin{pmatrix} n_j \\ m_{j-1} \end{pmatrix} := \begin{pmatrix} a_i n_j \\ (-1)^i a_i m_{j-1} \end{pmatrix}.$$

Lemma 18.6. *Let A be a DG R-algebra, let M and N be DG A-modules, and let $\alpha : M \to N$ be a morphism of DG A-modules. Then* $\operatorname{Cone}(\alpha)$ *is a DG A-module.*

Proof. We will check the Leibniz rule and leave the other details as an exercise for the reader.

Fix $a_i \in A_i$. Let $n_j \in N_j$ and $m_{j-1} \in M_{j-1}$. Then we have the following:

$$\begin{aligned}
\partial_{i+j}^{\operatorname{Cone}(\alpha)}\Big(a_i \begin{pmatrix} n_j \\ m_{j-1} \end{pmatrix}\Big) &= \partial_{i+j}^{\operatorname{Cone}(\alpha)} \begin{pmatrix} a_i n_j \\ (-1)^i a_i m_{j-1} \end{pmatrix} \\
&= \begin{pmatrix} \partial_{i+j}^N(a_i n_j) + \alpha_{i+j-1}((-1)^i a_i m_{j-1}) \\ -\partial_{i+j-1}^M((-1)^i a_i m_{j-1}) \end{pmatrix} \\
&= \begin{pmatrix} \partial_i^A(a_i) n_j + (-1)^i a_i \partial_j^N(n_j) + (-1)^i a_i \alpha_{j-1}(m_{j-1}) \\ (-1)^{i+1}\big(\partial_i^A(a_i) m_{j-1} + (-1)^i a_i \partial_{j-1}^M(m_{j-1})\big) \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
\partial_i^A(a_i) \begin{pmatrix} n_j \\ m_{j-1} \end{pmatrix} + (-1)^i a_i \partial_j^{\operatorname{Cone}(\alpha)} \begin{pmatrix} n_j \\ m_{j-1} \end{pmatrix} &= \begin{pmatrix} \partial_i^A(a_i) n_j \\ (-1)^{i-1} \partial_i^A(a_i) m_{j-1} \end{pmatrix} + (-1)^i a_i \begin{pmatrix} \partial_j^N(n_j) + \alpha_{j-1}(m_{j-1}) \\ -\partial_{j-1}^M(m_{j-1}) \end{pmatrix} \\
&= \begin{pmatrix} \partial_i^A(a_i) n_j + (-1)^i a_i \partial_j^N(n_j)(- 1)^i a_i \alpha_{j-1}(m_{j-1}) \\ (-1)^{i-1} \partial_i^A(a_i) m_{j-1} - (-1)^{2i} a_i \partial_{j-1}^M(m_{j-1}) \end{pmatrix}.
\end{aligned}$$

Since $i + 1 \equiv i - 1 \pmod 2$ and $i + 1 + i \equiv 2i + 1 \pmod 2$, we conclude that the Leibniz rule holds. □

Lemma 18.7. *Let A be a DG R-algebra, let M and N be DG A-modules, and let $\alpha : M \to N$ be a morphism over A. Then there exist morphisms of DG A-modules*

ϵ and τ such that

$$\epsilon : N \to \mathrm{Cone}(\alpha)$$
$$n \mapsto \begin{pmatrix} n \\ 0 \end{pmatrix}$$
$$\tau : \mathrm{Cone}(\alpha) \to \Sigma M$$
$$\begin{pmatrix} n \\ m \end{pmatrix} \mapsto m$$

and the sequence $0 \to N \xrightarrow{\epsilon} \mathrm{Cone}(\alpha) \xrightarrow{\tau} \Sigma M \to 0$ *is exact.*

Proof. We check that the maps ϵ and τ respect the A-module actions. Let $a \in A$ and $n \in N$. Then we have the following computations:

$$\epsilon(an) = \begin{pmatrix} an \\ 0 \end{pmatrix}$$
$$a\epsilon(n) = a \begin{pmatrix} n \\ 0 \end{pmatrix} = \begin{pmatrix} an \\ \pm a0 \end{pmatrix}.$$

Let $m \in M$. Then we have the following computations:

$$\tau(a \begin{pmatrix} n \\ m \end{pmatrix}) = \tau \begin{pmatrix} an \\ (-1)^{|a|}am \end{pmatrix} = (-1)^{|a|}am$$
$$a\tau \begin{pmatrix} n \\ m \end{pmatrix} = a * m = (-1)^{|a|}am.$$

□

19. Day (April 27th)

Let A be a DG R-algebra where R is a commutative noetherian ring. We start by describing the module structure of the following:

Let $A = \cdots \xrightarrow{\partial_2^A} A_1 \xrightarrow{\partial_1^A} A_0 \to 0$ be a commutative DG R-algebra where R is a commutative noetherian ring. Let $n \in \mathbb{Z}$, and let Λ be a set. Then $A^{(\Lambda)} = \coprod_{\lambda\in\Lambda} Ae_\lambda$ is a semifree DG A-module with basis $E = \{e_\lambda | \lambda \in \Lambda\}$. The shift of $A^{(\Lambda)}$ is defined to be $\Sigma^n A^{(\Lambda)} = \Sigma^n \coprod_{\lambda\in\Lambda} A * e_\lambda$. If $e_\lambda \in (\Sigma^n A^{(\Lambda)})_n$ and $a_i \in (A)_i$ (that is, in degree i), then $a_i * e_\lambda = (-1)^{in} a_i e_\lambda$. An arbitrary element in $\coprod_{\lambda\in\Lambda} A * e_\lambda$ is a finite sum. If $a_\lambda \in (A)_j$ for all λ and all $a_i \in A_i$, then $a_i * (\sum_{\lambda\in\Lambda}^{finite} a_\lambda * e_\lambda) = \sum_{\lambda\in\Lambda}^{finite} a_i * (a_\lambda * e_\lambda) = \sum_{\lambda\in\Lambda}^{finite} (a_i a_\lambda) * e_\lambda$, that is, the associative law holds. The differential satisfies the Leibniz rule and we have $\partial_{j+n}^{\Sigma^n A^{(\Lambda)}}(a_\lambda * e_\lambda) = \partial_j^A(a_\lambda) * e_\lambda + (-1)^j a_\lambda * \partial_n^{\Sigma^n A^{(\Lambda)}}(e_\lambda) = \partial_j^A(a_\lambda) * e_\lambda$.

Lemma 19.1. *Let $n \in \mathbb{Z}$, and let Λ be a set. Let L be a DG A-module and $X = \{x_\lambda \in L_n | \lambda \in \Lambda\} \subseteq \mathrm{Ker}(\partial_n^L) \subseteq L_n$. Then there exists a DG A-module morphism (that is, a degree zero homomorphism)*

$$\phi : \Sigma^n A^{(\Lambda)} \to L$$

where the basis elements $e_\lambda \mapsto x_\lambda$ for all $\lambda \in \Lambda$ and in general the map is defined by $\sum_{\lambda\in\Lambda}^{finite} a_\lambda * e_\lambda \mapsto \sum_{\lambda\in\Lambda}^{finite} a_\lambda x_\lambda$.

Proof. The map ϕ is well defined since the representations are unique, that is, $E = \{e_\lambda | \lambda \in \Lambda\}$ is a semibasis and this module is semifree; the details are left as an exercise for the reader.

We show that the necessary diagrams commute and have the necessary structure for the elements $a_\lambda * e_\lambda$. The case for the general elements $\sum_{\lambda\in\Lambda}^{finite} a_\lambda * e_\lambda$ is similar.

To show that the diagram

$$\begin{array}{ccc}
\left(\Sigma^N A^{(\Lambda)}\right)_{j+n} & \xrightarrow{\partial_{j+n}^{\Sigma^n A^{(\Lambda)}}} & \left(\Sigma^n A^{(\Lambda)}\right)_{j+n-1} \\
\Big\downarrow{\scriptstyle \phi_{j+n}} & & \Big\downarrow{\scriptstyle \phi_{j+n-1}} \\
L_{j+n} & \xrightarrow{\partial_{j+n}^{L}} & L_{j+n-1}
\end{array}$$

commutes, we have the following computations:

$$\begin{aligned}
\partial_{j+n}^L(\phi_{j+n}(a_\lambda * e_\lambda)) &= \partial_{j+n}^L(a_\lambda x_\lambda) \\
&= \partial_j^A(a_\lambda)x_\lambda + (-1)^j a_\lambda \partial_n^L(x_\lambda) \\
&= \partial_j^A(a_\lambda)x_\lambda \\
\phi_{j+n-1}(\partial_{j+n}^{\Sigma^n A^{(\Lambda)}}(a_\lambda * e_\lambda)) &= \phi_{j+n-1}(\partial_j^A(a_\lambda) * e_\lambda) \\
&= \partial_j^A(a_\lambda)x_\lambda.
\end{aligned}$$

Thus the diagram commutes. To show that ϕ is A-linear, we have the following computations:

$$\begin{aligned}
\phi_{i+j+n}(a_i * (a_\lambda * e_\lambda)) &= \phi_{i+j+n}((a_i a_\lambda) * e_\lambda) \\
&= (a_i a_\lambda)x_\lambda \\
&= a_i(a_\lambda x_\lambda) \\
&= a_i \phi_{j+n}(a_\lambda * e_\lambda).
\end{aligned}$$

Therefore ϕ is A-linear. □

The above Lemma is a primitive version of mapping a free module onto a module.

Lemma 19.2. *Let $\alpha : L \to M$ be a morphism of DG A-modules, and let $n \in \mathbb{Z}$. Then we have the following:*

(a) *There exist morphisms $\beta : \Sigma^n A^{(Y)} \to L$ and $\gamma : \mathrm{Cone}(\beta) \to M$ such that the diagram*

$$\begin{array}{ccc}
L & \xrightarrow{\alpha} & M \\
\Big\downarrow{\scriptstyle \epsilon} & \nearrow{\scriptstyle \gamma} & \\
\mathrm{Cone}(\beta) & &
\end{array}$$

commutes, $\mathrm{H}_n(\gamma)$ is injective, and $\alpha_i \cong \gamma_i$ for all $i < n$. That is, for each i, we have the diagram

$$\begin{array}{ccc}
L_i & \xrightarrow{\alpha_i} & M_i \\
{\scriptstyle \epsilon_i}\Big\downarrow{\scriptstyle \cong} & \nearrow{\scriptstyle \gamma_i} & \\
\mathrm{Cone}(\beta)_i & &
\end{array}$$

where ϵ_i is an isomorphism and hence $\alpha_i \cong \gamma_i$;

(b) *If* $\mathrm{H}_n(\alpha)$ *is onto, then* β *and* γ *can be chosen such that* $\mathrm{H}_n(\gamma)$ *is an isomorphism; and*
(c) *If* $\mathrm{Ker}(\mathrm{H}_n(\alpha))$ *is finitely generated over* $\mathrm{H}_0(A)$*, then* β *can be chosen such that* Y *is finite.*

Proof. We prove parts (a) and (b) together. Part (c) follows from the proof of part (a). Since $\mathrm{H}_n(\alpha) : \mathrm{H}_n(L) \to \mathrm{H}_n(M)$, we have that $\mathrm{Ker}(\mathrm{H}_n(\alpha)) \subseteq \mathrm{H}_n(L) = \mathrm{Ker}(\partial_n^L)/\,\mathrm{Im}(\partial_{n+1}^L)$. Choose $Y \subseteq \mathrm{Ker}(\partial_n^L)$ such that the set $\{\overline{y} \in \mathrm{H}_n(L) | y \in Y\}$ is a generating set for $\mathrm{Ker}(\mathrm{H}_n(\alpha))$ over $\mathrm{H}_0(A)$. Then Lemma 19.1 implies that there exists a morphism of DG A-modules $\beta : \Sigma^n A^{(Y)} \to L$ defined on its basis elements by $e_y \mapsto y$ and in general by $\sum_{y\in Y}^{finite} a_y * e_y \mapsto \sum_{y\in Y}^{finite} a_y y$. If $\overline{y} \in \mathrm{Ker}(\mathrm{H}_n(\alpha))$, then we have $\overline{\alpha_n(y)} = \mathrm{H}_n(\alpha)(\overline{y}) = 0$ where $\overline{\alpha_n(y)} \in \mathrm{H}_n(M) = \mathrm{Ker}(\partial_n^M)/\,\mathrm{Im}(\partial_{n+1}^M)$. This implies that $\alpha_n(y) \in \mathrm{Im}(\partial_{n+1}^M)$, say $\alpha_n(y) = \partial_{n+1}^M(m_y)$ where $m_y \in M_{n+1}$.

Define the map $\gamma : \mathrm{Cone}(\beta) \to M$ by $\gamma_p : \mathrm{Cone}(\beta)_p \to M_p$ where $\mathrm{Cone}(\beta)_p = L_p \oplus (\Sigma^n A^{(Y)})_{p-1}$. Recall that

$$a_i \begin{pmatrix} l_p \\ a_\lambda * e_\lambda \end{pmatrix} = \begin{pmatrix} a_i l_p \\ (-1)^i a_i * (a_\lambda * e_\lambda) \end{pmatrix}.$$

We define the map γ_p by

$$\gamma_p \begin{pmatrix} l_p \\ \sum_{y\in Y}^{finite} a_y * e_y \end{pmatrix} = \alpha_p(l_p) + (-1)^{p-n-1} \sum_{y\in Y}^{finite} a_y m_y$$

where the element a_y is in degree $p-n-1$, the element e_y is in degree n, and the element m_y is in degree $n+1$. It is left as an exercise for the reader to check that γ is a morphism of DG A-modules.

Then we have the following commutative diagram:

$$\begin{array}{ccc} L_p & \xrightarrow{\alpha_p} & M_p \\ {\scriptstyle \epsilon_p}\big\downarrow & \nearrow_{\gamma_p} & \\ \mathrm{Cone}(\beta)_p & & \end{array}$$

and for $l_p \in L_p$ we have $\gamma_p(\epsilon_p(l_p)) = \gamma_p(\begin{pmatrix} l_p \\ 0 \end{pmatrix}) = \alpha_p(l_p)$.

We have the short exact sequence

$$0 \to L \to \mathrm{Cone}(\beta) \to \Sigma^{n+1} A^{(Y)} \to 0.$$

Part of the long exact sequence associated to the short exact sequence above is

$$\cdots \to \mathrm{H}_{n+1}(\Sigma^{n+1} A^{(Y)}) \xrightarrow{\Delta_{n+1}} \mathrm{H}_n(L) \xrightarrow{\mathrm{H}_n(\epsilon)} \mathrm{H}_n(\mathrm{Cone}(\beta)) \to \mathrm{H}_n(\Sigma^{n+1} A^{(Y)}) \to \cdots.$$

Since $\mathrm{H}_n(\Sigma^{n+1}A^{(Y)}) = 0$, we can rewrite the sequence as

$$\begin{array}{ccccccc}
\cdots \longrightarrow & \mathrm{H}_{n+1}(\Sigma^{n+1}A^{(Y)}) & \xrightarrow{\Delta_{n+1}} & \mathrm{H}_n(L) & \xrightarrow{\mathrm{H}_n(\epsilon)} & \mathrm{H}_n(\mathrm{Cone}(\beta)) & \longrightarrow 0 \\
& \cong\downarrow & & \downarrow \mathrm{H}_n(\alpha) & \swarrow \mathrm{H}_n(\gamma) & & \\
& \mathrm{H}_0(A^{(Y)}) & & \mathrm{H}_n(M) & & & \\
& \cong\downarrow & & & & & \\
& \mathrm{H}_0(A)^{(Y)}. & & & & &
\end{array}$$

Then Δ_{n+1} maps onto $\mathrm{Ker}(\mathrm{H}_n(\alpha))$. Since $\mathrm{H}_n(\epsilon)$ is onto, a diagram chase shows that $\mathrm{H}_n(\gamma)$ is injective. Also, if $\mathrm{H}_n(\alpha)$ is onto, then a diagram chase shows that $\mathrm{H}_n(\gamma)$ is onto.

For $i < n$, we have

$$\begin{array}{ccccccc}
0 = \mathrm{H}_{i+1}(\Sigma^{n+1}A^{(Y)}) & \longrightarrow & \mathrm{H}_i(L) & \xrightarrow{\cong} & \mathrm{H}_i(\mathrm{Cone}(\beta)) & \longrightarrow & \mathrm{H}_i(\Sigma^{n+1}A^{(Y)}) = 0 \\
& & \downarrow \mathrm{H}_i(\alpha) & \swarrow \mathrm{H}_i(\gamma) & & & \\
& & \mathrm{H}_i(M) & & & &
\end{array}$$

where the isomorphism is a consequence of the exactness of the sequence. Therefore $\mathrm{H}_i(\gamma) = \mathrm{H}_i(\alpha)$. That is, if $\mathrm{H}_i(\alpha)$ is an isomorphism for all $i < n$, then so is $\mathrm{H}_i(\gamma)$. □

20. Day May 4th

For this section let A and B be DG R-algebras where R is a commutative local noetherian ring.

Lemma 20.1. *Let M be a DG A-module, $n \in \mathbb{Z}$, and $X \in \mathrm{Ker}(\partial_n^M)$. Then there exists a morphism of DG A-modules*

$$\phi : \Sigma^n A^{(X)} \to M$$

defined by $\sum_{x\in X}^{\textit{finite}} a_x e_x \mapsto \sum_{x\in X}^{\textit{finite}} a_x x$ where e_x denotes the basis elements. Furthermore, $\mathrm{H}_n(\phi) : \mathrm{H}_n(\Sigma^n A^{(X)}) \to \mathrm{H}_n(M)$ and $\mathrm{Im}(\mathrm{H}_n(\phi))$ is a submodule of $\mathrm{H}_n(M)$ generated over $\mathrm{H}_0(A)$ by $\overline{X} \subseteq \mathrm{H}_n(M)$.

Proof. This follows from the equalities

$$\mathrm{H}_n(\phi)(\overline{\sum_{x\in X}^{\text{finite}} a_x e_x}) = \overline{\sum_{x\in X}^{\text{finite}} a_x e_x} = \sum_{x\in X}^{\text{finite}} \overline{a_x x}.$$

□

Proposition 20.2. *Let M be a DG A-module. Then we have the following:*

(a) *There exists a morphism of DG A-modules*

$$\alpha : \coprod_{n\in\mathbb{Z}} \Sigma^n A^{(X_n)} \to M$$

such that $\mathrm{H}_m(\alpha)$ is onto for all m;

(b) *If $\mathrm{H}_m(M) = 0$ for all $m < u$, then α can be chosen such that $X_m = \emptyset$ for all $m < u$; and*

(c) *If* $\mathrm{H}_m(M)$ *is finitely generated over* $\mathrm{H}_0(A)$ *for all* m*, then* α *can be chosen such that* X_m *is finite for all* m*.*

Proof. We have $\alpha = \coprod_{n\in\mathbb{Z}} \phi_n : \coprod_{n\in\mathbb{Z}} \Sigma^n A^{(X_n)} \to M$ where $X_m \subseteq \mathrm{Ker}(\partial_m^M)$ is the generating set for $\mathrm{H}_m(M)$ over $\mathrm{H}_0(A)$. The details of this proof are left as an exercise for the reader. □

Theorem 20.3. *Let* M *be a DG* B*-module such that* $\mathrm{H}_i(M) = 0$ *for all* $i < u$*. Then we have the following:*

(a) M *has a semifree resolution over* A *of* $\beta : F \xrightarrow{\simeq} M$ *such that* $F_i = 0$ *for all* $i < u$*; and*
(b) *If* $\mathrm{H}_0(A)$ *is noetherian and for all* i *and the modules* $\mathrm{H}_i(A)$ *and* $\mathrm{H}_i(M)$ *are finitely generated over* $\mathrm{H}_0(A)$*, then there exists a semifree resolution* $F \xrightarrow{\simeq} M$ *such that* $F_i = 0$ *for all* $i < u$ *and a semibasis* $E \subseteq F$ *that satisfies* $|E \cap F_i| < \infty$ *for all* i*.*

Proof. We show the proof for part (a) and note that part (b) follows from a similar argument.

Let $\alpha : \coprod_{n\in\mathbb{Z}} \Sigma^n A^{(X_n)} \to M$ such that $\mathrm{H}_m(\alpha)$ is onto for all m. Proposition 20.2(b) implies that we can write α as the following

$$\alpha : \coprod_{n\in\mathbb{Z}, n\geqslant u} \Sigma^n A^{(X_n)} \to M.$$

We claim that for all $n \in \mathbb{Z}$ there exist morphisms $\alpha^{(n)} : L^{(n)} \to M$ and $\epsilon^{(n-1)} : L^{(n-1)} \to L^{(n)}$ such that:

(1) $L^{(n)}$ is semifree for all n;
(2) $\mathrm{H}_i(\alpha^{(n)}) : \mathrm{H}_i(L^{(n)}) \to \mathrm{H}_i(M)$ is an isomorphism for all $i \leqslant n$;
(3) The following diagram commutes

$$\begin{array}{ccc} L^{(n-1)} & \xrightarrow{\alpha^{(n-1)}} & M \\ {\scriptstyle \epsilon^{(n-1)}}\downarrow & \nearrow{\scriptstyle \alpha^{(n)}} & \\ L^{(n)}; & & \end{array}$$

(4) $\epsilon_i^{(n)}$ is injective for all i and all n; and
(5) $\epsilon_i^{(n-1)}$ is an isomorphism for all $i < n$.

A note about the notation here: the subscripts indicate the degree of the map or module and the superscripts in parenthesis denote the nth step of the claim.

For all $n < u$, set $\alpha^{(n)} = \alpha$, set $\epsilon^{(n-1)} = id_L$ and set $L = L^{(n)} = \coprod_{n\geqslant u} \Sigma^n A^{(X_n)}$. We proceed by induction on n. Since u is fixed, our base case is $n < u$. The inductive step follows from Lemma 19.2(b) with $\alpha^{(n)} = \gamma$, $L^{(n)} = \mathrm{Cone}(\beta)$, and $\epsilon^{(n-1)} = \epsilon$.

We wave our arms at the finitely generated part, noetherianness is crucial here since this implies that the kernels of the maps are finitely generated (submodules of finitely generated modules are finitely generated).

Note $\epsilon_i^{(k)}$ is an isomorphism for all $i < k+1$. Let $E^{(n)}$ be a semibasis for $L^{(n)}$. Let $F = \varinjlim_n L^{(n)}$. This is a proof from Apassov. At this point he drew a box and called it a day. The details are below.

We have the commutative diagram

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\epsilon_n^{(n-2)}} & L_n^{(n-1)} & \xrightarrow{\epsilon_n^{(n-1)}} & L_n^{(n)} & \xrightarrow{\epsilon_n^{(n)}} & L_n^{(n+1)} & \xrightarrow{\epsilon_n^{(n+1)}} & \cdots \\ & & & \searrow^{\alpha_n^{(n-1)}} & \downarrow{\scriptstyle \alpha_n^{(n)}} & \swarrow_{\alpha_n^{(n+1)}} & & & \\ \cdots & & & & M_n & & & & \cdots . \end{array}$$

Therefore $\epsilon_n^{(n)}$ is an isomorphism and $\epsilon_n^{(j)}$ is an isomorphism for all $j > n$, that is, stabilization occurs. So $F_n = L_n^{(n)} \cong L_n^{(n+1)} \cong \cdots$. Then $\beta_n = \alpha_n^{(n)} \cong \alpha_n^{(n+1)} \cong \cdots$.

To show that F is semifree we set $G_n = E_n^{(n)} \cong E_n^{(n+1)} \cong \cdots$. Let $x \in F_n$, that is, $x \in L_n^{(n)}$. Then $L_n^{(n)}$ is semifree with semibasis $E^{(n)}$. Therefore x is uniquely a linear combination of elements of $E^{(n)}$ with coefficients in A because $A_i = 0$ for all $i < 0$ and $x \in L_n^{(n)}$, the only basis vector in linear combination are in $E_i^{(n)}$ with $i \leqslant n$. Therefore it is a unique linear combination in L.

We have $F_i = 0$ for all $i < u$ from the construction $L_i^{(n)} = 0$ for all $i < u$.

We have for all i, the map $\mathrm{H}_i(\beta)$ is an isomorphism because $\mathrm{H}_i(\beta) \cong \mathrm{H}_i(\alpha_n^{(n)})$ is an isomorphism.

The map β is DG A-linear because $\beta_i = \alpha_i^{(k)}$ is DG A-linear. □

We can extend these concepts to construct Ext_A and Tor^A.

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