<u>SUBSPACE TEST</u>

<u>Strategy: We want to see if H is a subspace of V.</u>

1.) Is the zero vector of V also in H? If no, then H is not a subspace of V. If yes, then move on to step 2.

2.) Identify c, $\vec{u}$, $\vec{v}$, and list any “facts”.

3.) Is $\vec{u}+\vec{v}$ in H? If yes, then move on to step 4. If no, then give a specific example to show that $\vec{u}$ and $\vec{v}$ are in H but $\vec{u}+\vec{v}$ is not in H. This shows that H is not a subspace of V.

4.) Is $c\vec{u}$ in H? If yes, then H is a subspace of V. If no, then give a specific example to show that $\vec{u}$ is in H but $c\vec{u}$ is not in H. This shows that H is not a subspace of V.

Note: If H is defined with the variables u, v, or c, feel free to use different letters. For example, let your scalar be d instead of c or let your two vectors be g and h instead of u and v. There is nothing special about using c, u, and v. Just be sure to be consistent. Don't start using d as your scalar and end up using c as your scalar.

Example 1

Let $H = \left\{ \begin{pmatrix} a+b+c \\ 2a+3b \\ 4c \end{pmatrix} \middle| a,b,c \in R \right\}$. Is H a subspace of R^3?

1.) Is the zero vector of R^3 also in H?

We need to see if the equation $\begin{pmatrix} a+b+c \\ 2a+3b \\ 4c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a solution. A solution to this equation is $a = b = c = 0$. So $\vec{0}$ is in H.

2.) Identify d, $\vec{u}$, $\vec{v}$, and list any "facts". (I'm going to use d as my scalar because c has already been used.)

$$d \in R, \ \vec{u} = \begin{pmatrix} u_1 + u_2 + u_3 \\ 2u_1 + 3u_2 \\ 4u_3 \end{pmatrix}, \text{ and } \vec{v} = \begin{pmatrix} v_1 + v_2 + v_3 \\ 2v_1 + 3v_2 \\ 4v_3 \end{pmatrix}$$

3.) Is $\vec{u} + \vec{v}$ in H?

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + u_2 + u_3 + v_1 + v_2 + v_3 \\ 2u_1 + 3u_2 + 2v_1 + 3v_2 \\ 4u_3 + 4v_3 \end{pmatrix} = \begin{pmatrix} (u_1 + v_1) + (u_2 + v_2) + (u_3 + v_3) \\ 2(u_1 + v_1) + 3(u_2 + v_2) \\ 4(u_3 + v_3) \end{pmatrix}$$

Here $a = u_1 + v_1$, $b = u_2 + v_2$, and $c = u_3 + v_3$. So $\vec{u} + \vec{v}$ is in H.

4.) Is $d\vec{u}$ in H?

$$d\vec{u} = \begin{pmatrix} d(u_1 + u_2 + u_3) \\ d(2u_1 + 3u_2) \\ d(4u_3) \end{pmatrix} = \begin{pmatrix} du_1 + du_2 + du_3 \\ 2du_1 + 3du_2 \\ 4du_3 \end{pmatrix}$$

Here $a = du_1$, $b = du_2$, and $c = du_3$. So $d\vec{u}$ is in H. This means that H is a subspace of R^3.

Example 2

Let $W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} : c = a + 2b \ \ \& \ \ d = a - 3b \right\}$. Is W a subspace of R^4?

1.) Is the zero vector of R^4 also in W?

We need to see if the equation $\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} a \\ b \\ a+2b \\ a-3b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ has a solution.

A solution to this equation is $a = b = 0$. So $\vec{0}$ is in W.

2.) Identify k, $\vec{u}$, $\vec{v}$, and list any "facts".

$$k \in R, \ \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{pmatrix}, \ \vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{pmatrix}$$

Our "facts" are $u_3 = u_1 + 2u_2$, $u_4 = u_1 - 3u_2$, $v_3 = v_1 + 2v_2$, and $v_4 = v_1 - 3v_2$.

3.) Is $\vec{u} + \vec{v}$ in W?

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ u_4 + v_4 \end{pmatrix}$$

Here $a = u_1 + v_1$, $b = u_2 + v_2$, $c = u_3 + v_3$, and $d = u_4 + v_4$. We need to know if $c = a + 2b$ and $d = a - 3b$.

$$c = u_3 + v_3 = (u_1 + 2u_2) + (v_1 + 2v_2) = (u_1 + v_1) + 2(u_2 + v_2) = a + 2b$$
$$d = u_4 + v_4 = (u_1 - 3u_2) + (v_1 - 3v_2) = (u_1 + v_1) - 3(u_2 + v_2) = a - 3b$$

So $\vec{u} + \vec{v}$ is in W.

4.) Is $k\vec{u}$ in W?

$$k\vec{u} = \begin{pmatrix} ku_1 \\ ku_2 \\ ku_3 \\ ku_4 \end{pmatrix}$$

Here $a = ku_1$, $b = ku_2$, $c = ku_3$, and $d = ku_4$. We need to know if $c = a + 2b$ and $d = a - 3b$.

$c = ku_3 = k(u_1 + 2u_2) = ku_1 + 2ku_2 = a + 2b$ and $d = ku_4 = k(u_1 - 3u_2) = ku_1 - 3ku_2 = a - 3b$

So $k\vec{u}$ is in W. This means that W is a subspace of R^4.

Example 3

Let $H = \left\{ \begin{pmatrix} 2a + b \\ 2b + 4c - 1 \\ a + b + c \end{pmatrix} \middle| \ a, \ b, \ c \in R \right\}$. Is H a subspace of R^3 ?

1.) Is the zero vector of R^3 also in H?

Does the equation $\begin{pmatrix} 2a + b \\ 2b + 4c - 1 \\ a + b + c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ have a solution? No, this equation can be written as the

system $\begin{matrix} 2a + b = 0 \\ 2b + 4c = 1 \\ a + b + c = 0 \end{matrix}$. Looking at the augmented matrix $\left(\begin{array}{ccc|c} 2 & 1 & 0 & 0 \\ 0 & 2 & 4 & 1 \\ 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right)$ we can see

that this system has no solution. This means that $\vec{0}$ is not in H. So H is not a subspace of R^3 .

Example 4

Let $H = \left\{ \begin{pmatrix} abc \\ 2a+3b \\ 4c \end{pmatrix} \middle| a,b,c \in R \right\}$. Is H a subspace of R^3?

1.) Is the zero vector of R^3 also in H?

We need to see if the equation $\begin{pmatrix} abc \\ 2a+3b \\ 4c \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ has a solution. A solution to this equation is $a = b = c = 0$. So $\vec{0}$ is in H.

2.) Identify d, $\vec{u}$, $\vec{v}$, and list any "facts". (I'm going to use d as my scalar because c has already been used.)

$$d \in R,\ \vec{u} = \begin{pmatrix} u_1 * u_2 * u_3 \\ 2u_1 + 3u_2 \\ 4u_3 \end{pmatrix},\ \text{and}\ \vec{v} = \begin{pmatrix} v_1 * v_2 * v_3 \\ 2v_1 + 3v_2 \\ 4v_3 \end{pmatrix}$$

3.) Is $\vec{u} + \vec{v}$ in H?

$$\vec{u} + \vec{v} = \begin{pmatrix} u_1 * u_2 * u_3 + v_1 * v_2 * v_3 \\ 2u_1 + 3u_2 + 2v_1 + 3v_2 \\ 4u_3 + 4v_3 \end{pmatrix} = \begin{pmatrix} u_1 * u_2 * u_3 + v_1 * v_2 * v_3 \\ 2(u_1 + v_1) + 3(u_2 + v_2) \\ 4(u_3 + v_3) \end{pmatrix}$$

Looking at the last two entries we see that $a = u_1 + v_1$, $b = u_2 + v_2$, and $c = u_3 + v_3$.

However, $abc = (u_1 + v_1)(u_2 + v_2)(u_3 + v_3) \neq u_1 * u_2 * u_3 + v_1 * v_2 * v_3$, in general.

For example, let $u_1 = u_2 = u_3 = 2$, $v_2 = 3$, and $v_1 = v_3 = -3$. So $\vec{u} = \begin{pmatrix} 8 \\ 10 \\ 8 \end{pmatrix}$, $\vec{v} = \begin{pmatrix} 27 \\ 3 \\ -12 \end{pmatrix}$, and

$\vec{u} + \vec{v} = \begin{pmatrix} 35 \\ 13 \\ -4 \end{pmatrix} = \begin{pmatrix} abc \\ 2a+3b \\ 4c \end{pmatrix}$. Now we have that $c = -1$, $a = \frac{13-3b}{2}$, and $\left(\frac{13-3b}{2}\right) * b * (-1) = 35$

or $3b^2 - 13b - 70 = 0$. Using the quadratic formula, we get $b = \frac{13 \pm \sqrt{169-840}}{6} \notin R$. So, in general, $\vec{u} + \vec{v}$ is not in H. This means that H is not a subspace of R^3.

EXERCISES

For problems 1-5, determine if H is a subspace of R^n. If so, write it as the span of a set of vectors in R^n and the column space of a matrix.

1.) $H = \left\{ \begin{pmatrix} a+b+2c \\ ab+c \end{pmatrix} \middle| \ a,\ b,\ c \in R \right\}$

2.) $H = \left\{ \begin{pmatrix} a+2b \\ a+b+c \\ 0 \end{pmatrix} \middle| \ a,\ b,\ c \in R \right\}$

3.) $H = \left\{ \begin{pmatrix} x+2y-z \\ 4y+6z \\ 7x-3y+14z \end{pmatrix} \middle| \ x,\ y,\ z \in R \right\}$

4.) $H = \left\{ \begin{pmatrix} 2x+3y+4z+6w \\ y-z \\ 1 \\ x+y-z-w \end{pmatrix} \middle| \ x,\ y,\ z,\ w \in R \right\}$

5.) $H = \left\{ \begin{pmatrix} a-b+7c \\ 3a-4b+8c-9d \\ d-c \\ 6a-17d \end{pmatrix} \middle| \ a,\ b,\ c,\ d \in R \right\}$

6.) Let $A = \begin{pmatrix} 1 & 2 & 4 & 6 \\ -1 & 5 & -3 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$. Is the solution set of $A\vec{x} = \vec{0}$ as subspace of R^3? Is the solution set of $A\vec{x} = \vec{0}$ as subspace of R^4?

7.) Let $A = \begin{pmatrix} 1 & 2 & 4 & 6 \\ -1 & 5 & -3 & 2 \\ 1 & 2 & 3 & 4 \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}$. Is the solution set of $A\vec{x} = \vec{b}$ as subspace of R^3? Is the solution set of $A\vec{x} = \vec{b}$ as subspace of R^4?

For problems 8-14, determine if W is a subspace of R^n.

8.) $W = \left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \middle| \ abc = 0 \right\}$

9.) $W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ x+y+z = 0 \right\}$

10.) $W = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \ \sin x = \frac{\pi}{2} \right\}$

11.) $W = \left\{ \begin{pmatrix} r \\ s \\ t \end{pmatrix} \middle| \ r+s \leq t \right\}$

12.) $W = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} \middle| \ 2x+3y = z+w \right\}$

13.) $W = \left\{ \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} \middle| \ a+b = 3c \text{ and } b+c = 4d \right\}$

14.) Let $T : R^n \to R^m$ be a linear transformation. Determine if $W = \left\{ \vec{x} \ \middle| \ \vec{x} \in R^n \text{ and } T(\vec{x}) = \vec{0} \right\}$ is a subspace of R^n.

ANSWERS TO THE ODD EXERCISES

1.) H is not a subspace of R^2.

3.) H is a subspace of R^3 where $H = \text{span}\left\{\begin{pmatrix}1\\0\\7\end{pmatrix}, \begin{pmatrix}2\\4\\-3\end{pmatrix}, \begin{pmatrix}-1\\6\\14\end{pmatrix}\right\} = \text{Col}A$ with $A = \begin{pmatrix}1 & 2 & -1\\0 & 4 & 6\\7 & -3 & 14\end{pmatrix}$.

5.) H is a subspace of R^4 where $H = \text{span}\left\{\begin{pmatrix}1\\3\\0\\6\end{pmatrix}, \begin{pmatrix}-1\\-4\\0\\0\end{pmatrix}, \begin{pmatrix}7\\8\\-1\\0\end{pmatrix}, \begin{pmatrix}0\\-9\\1\\-17\end{pmatrix}\right\} = \text{Col}A$ with $A = \begin{pmatrix}1 & -1 & 7 & 0\\3 & -4 & 8 & -9\\0 & 0 & -1 & 1\\6 & 0 & 0 & -17\end{pmatrix}$.

7.) The solution set of $A\vec{x} = \vec{b}$ is not a subspace of R^3 or R^4.

9.) W is a subspace of R^3.

11.) W is not a subspace of R^3.

13.) W is a subspace of R^4.