Problem Set 1
Due: Thursday, September 12

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined in the *Problem Set Guidelines Sheet*.

Unless otherwise stated, all problems can be found in the appropriate *Problems* section of the textbook (*Elementary Number Theory* by U. Dudley, 2nd Edition).

(1) Section 2 # 5

(2) Section 2 # 9

(3) *Twin primes* are pairs of consecutive odd integers $(p, p+2)$ which are both prime. For example, $(3,5),(5,7),(11,13)$ are all pairs of twin primes. The *Twin Primes Conjecture* (still unresolved) claims that there are infinitely many pairs of twin primes.

   (a) Give three more examples of twin primes in addition to what is given in this exercise.

   (b) Building on this idea, we can define a set of "triplet primes" to be three consecutive odd integers $(p, p+2, p+4)$ which are all prime. A quick scan of our list of primes less than 100 shows that there is exactly one set of triplet primes in that range: namely $(3,5,7)$. Are there any other triplet primes (ever)? If so, give an example. If not, prove that there are no others.

(4) *Mersenne Primes:* A prime integer which is of the form $2^n - 1$ for some integer n is called a *Mersenne prime*. Mersenne primes are named after the French monk Marin Mersenne (1588 - 1648) who studied them. It is still unknown whether there are infinitely may Mersenne primes.

   (a) Find four Mersenne primes.

   (b) Let p be a prime integer. Must $2^p - 1$ be a prime integer also? If yes, prove your claim. If no, give a specific counter-example.

   (c) Preparation for part (d): Let x and y be integers. Recall the well-known formulas: $x^2 - y^2 = (x-y)(x+y)$ and $x^3 - y^3 = (x-y)(x^2+xy+y^2)$. Prove that for any integer $n \geq 1$

$$x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + xy^{n-2} + y^{n-1}).$$

   (*Note:* You do not need to use induction to prove this. Rather, try to simplify the right-hand side of the equation.)

   (d) Prove that if $2^n - 1$ is a Mersenne prime, then n must also be a prime integer. (*Hint:* Note that $x^{cd} - 1 = (x^c)^d - 1^d$ for any integer x and positive integers c, d.)

(5) *The E-zone:* The set consisting of the positive even integers along with 1, i.e., $E = \{1, 2, 4, 6, 8, 10, \ldots\}$, is called the *E-zone*. Any integer in the E-zone is called an *E-number*. We say that an E-number x divides an E-number y in the E-zone if there exists an E-number u such that $y = xu$. We call an E-number an *E-prime* if it is greater than 1 and its only E-zone divisors are 1 and itself. For example, $2, 6, 10, 14$ and 18 are the first few E-primes.

(a) Prove that an E-number is an E-prime if and only if it is twice an odd integer. (*Note:* There are two things to be shown here.)

(b) Let p be an E-prime and let a and b be E-numbers. Suppose that p divides ab in the E-zone. Must p divide a or b in the E-zone? If yes, provide a proof. If no, give a specific counter-example.

(c) Prove that every E-number greater than 1 can be factored into a product of E-primes.

(d) Find all possible E-prime factorizations of 36, 60, 72, and 360.

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(e) Although part (4c) shows the *existence* of E-prime factorization of E-numbers, you should have noticed that part (4d) shows that this factorization is not always *unique*. However, you should have noticed that the *number* of E-primes occurring in every factorization of a given E-number was always the same. For example, every factorization you found of 36 should have involved 2 E-primes, and every factorization you found of 72 should have involved 3 E-primes. Is this always the case? In other words, either prove or disprove (by giving a counter-example) the following conjecture:

Conjecture: Suppose the E-number n can be factored as a product of k E-primes. Then *every* factorization of n into E-primes involves exactly k E-prime factors.