# Problem Set 1
## Due: At the Beginning of Class on Thursday, January 23

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets*.

**Background Material:** The exercises in this section are based on prerequisite material for MTH 523. You may need to review Chapter 0 in the text to complete these exercises.

(1) (Gallian, Chapter 0 Exercises, #3) Determine 51 mod 13, 342 mod 85, 62 mod 15, 10 mod 15, $(82 \cdot 73)$ mod 7, $(51 + 68)$ mod 7, $(35 \cdot 24)$ mod 11, and $(47 + 68)$ mod 11.

(2) (Gallian, Chapter 0 Exercises, #13) Suppose that $m$ and $n$ are relatively prime and $r$ is any integer. Show that there are integers $x$ and $y$ such that $mx + ny = r$.

(3) Use mathematical induction to prove that $2^{n-1} \leq n!$ for every non-negative integer $n$. [Recall that $0! = 1$ and for $n > 0$, $n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n$.] *Hint:* You can start by verifying two base cases, namely $n = 0$ and $n = 1$.

(4) Let $S = \mathbb{R}^2$ be the set of ordered pairs of real numbers and, for $(a, b)$, $(c, d) \in S$, define $(a, b) \sim (c, d)$ if $5a + 9b = 5c + 9d$. Show that $\sim$ is an equivalence relation on $S$. There is a nice geometric description of the equivalence classes. What is it?

**New Material:** The exercises in this section are based on new material from our first week of classes.

(1) (Gallian, Chapter 1 Exercises, #1) With pictures and words, describe each symmetry in $D_3$ (the set of symmetries of an equilateral triangle).

(2) (Gallian, Chapter 1 Exercises, #2) Write out a complete Cayley table for $D_3$. Is $D_3$ Abelian? Support your answer with either a proof or a specific counter-example.

(3) (Gallian, Chapter 2 Exercises, #4) Which of the following sets are closed under the given operation?

  (a) $S = \{0, 4, 8, 12\}$ under addition mod 16
  (b) $S = \{0, 4, 8, 12\}$ under addition mod 15
  (c) $S = \{1, 4, 7, 13\}$ under multiplication mod 15
  (d) $S = \{1, 4, 5, 7\}$ under multiplication mod 9

(4) (Gallian, Chapter 2 Exercises, #12) Give an example of group elements $a$ and $b$ with the property that $a^{-1}ba \neq b$. *Hint:* Look at the group $D_4$.

(5) (Gallian, Chapter 2 Exercises, #22) Let $G$ be a group with the property that for any $x, y, z$ in the group, $xy = zx$ implies $y = z$. Prove that $G$ is Abelian.

(6) (Gallian, Chapter 2 Exercises, #52) Let $G = \left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} : a \in \mathbb{R}, a \neq 0 \right\}$. Show that $G$ is a group under matrix multiplication. (Notice that each element of $G$ has an inverse even though the matrices have determinant 0!)

**Extra Credit GAP Exercise:** This next section is not required. However, you may choose to complete the tasks for extra credit on this Problem Set. In order to receive credit, you must use GAP.

Start by reading through Chapter $-1$ and Chapter 0 (but stop at the bottom of page 10) of the lab manual (`http://math.slu.edu/~rainbolt/FullManual8th.pdf`). You do not need to turn anything in from these chapters, but you'll probably want to do some of the problems just to get a feel for how to use GAP. Then do (to turn in) the following problem:

(1) As mentioned in the lab manual, there is a nice combinatorial formula for the sum of the first $n$ integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}.$$

There is a similarly nice formula for the sum of the first $n$ cubes. The goal of this problem is to use GAP to conjecture the formula, which you will then prove using induction.

(a) We can define a function in GAP that, given a positive integer $n$, outputs the sum of the first $n$ cubes. For example, if your function is called $f$, then $f(1) = 1^3 = 1$, $f(2) = 1^3 + 2^3 = 9$ and $f(3) = 1^3 + 2^3 + 3^3 = 36$. To do this, we need to use the GAP command "Sum". You can find out how to use this command by typing `?Sum` at a GAP prompt. To define this function, type into GAP

```
sumcubes := n->Sum([1..n], x->x^3);
```

at the command prompt. The output should yield

```
gap> sumcubes := n->Sum([1..n], x->x^3);
function( n ) ... end
```

Then, for example, to find $f(2) = 1^3 + 2^3$, type

```
sumcubes(2);
```

(b) Compute, using your function `sumcubes` and GAP, the sum of the first $n$ cubes for at least 4 different values of $n \geq 4$. Then use this output to conjecture a nice formula for the sum of the first $n$ cubes.

(c) Test your conjecture on a value of $n$ that is larger than any of the examples you previously computed.

(d) Use induction to prove that your conjecture holds for all integers $n \geq 1$.

Some Comments:

- Please intersperse your GAP commands and output with your explanations. You'll want to create a log file as described at `http://www.gap-system.org/Faq/faq.html#5.4`. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions.

- In general, be sure to read the questions in the lab manual carefully. Many contain multiple parts that aren't separated out as such.