Problem Set 2

Due: At the Beginning of Class on Thursday, February 6

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets.*

(1) (Gallian, Chapter 3 Exercises, #15) If a is an element of a group G and $|a| = 7$, show that a is the cube of some element of G.

(2) (Gallian, Chapter 3 Exercises, #16) Suppose that H is a nonempty subset of a group G with the property that if a and b belong to H then $a^{-1}b^{-1}$ belongs to H. Prove or disprove (with a specific example) that this is enough to guarantee that H is a subgroup of G.

(3) (Gallian, Chapter 3 Exercises, #20) Let x belong to a group. If $x^2 \neq e$ and $x^6 = e$, prove that $x^4 \neq e$ and $x^5 \neq e$. What can we say about the order of x?

(4) (Gallian, Chapter 3 Exercises, #21) Show that if a is an element of a group G, then $|a| \leq |G|$.

(5) (Gallian, Chapter 3 Exercises, #32) If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G.

(6) (Gallian, Chapter 3 Exercises, #42) If H is a subgroup of G, then by the *centralizer* $C(H)$ of H we mean the set $C(H) = \{x \in G \mid xh = hx \text{ for all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

(7) (Gallian, Chapter 3 Exercises, #50) Suppose a and b are group elements such that $|a| = 2, b \neq e$, and $aba = b^2$. Determine $|b|$.

(8) (Gallian, Chapter 3 Exercises, #52) Consider the elements $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ from $SL(2, \mathbb{R})$. Find $|A|, |B|$, and $|AB|$. Does your answer surprise you?

(9) (Gallian, Chapter 3 Exercises, #62) Compute the orders of the following groups.

(a) $U(3), U(4), U(12)$
(b) $U(5), U(7), U(35)$
(c) $U(4), U(5), U(20)$
(d) $U(3), U(5), U(15)$

On the basis of your answers, make a conjecture about the relationships among $|U(r)|, |U(s)|$, and $|U(rs)|$.

(10) (Gallian, Chapter 3 Exercises, #68) Let $H = \{A \in GL(2, \mathbb{R}) \mid \det(A) \text{ is an integer power of } 2\}$. Show that H is a subgroup of $GL(2, \mathbb{R})$.

(11) (Gallian, Chapter 3 Exercises, #70) Let G be a group of functions from $\mathbb{R}$ to $\mathbb{R}^*$, where the operation of G is multiplication of functions. Let $H = \{f \in G \mid f(2) = 1\}$. Prove that H is a subgroup of G. Can 2 be replaced by any real number?

(12) (Gallian, Supplementary Exercises for Chapters 1–4, #2) Let G be a group and let H be a subgroup of G. For any fixed x in G, define $xHx^{-1} = \{xhx^{-1} \mid h \in H\}$. Also, define $N(H) = \{x \in G \mid xHx^{-1} = H\}$. Prove that $N(H)$ (called the *normalizer* of H) is a subgroup of G.

Extra Credit GAP Exercises: This next section is not required. However, you may choose to complete the tasks for extra credit on this Problem Set. In order to receive credit, you must use GAP.

Please intersperse your GAP commands and output with your explanations. As usual, you should create a log file. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions. The GAP lab manual can be found at (`http://math.slu.edu/~rainbolt/FullManual8th.pdf`.

(1) (Lab Manual, Chapter 2 Exercises, #2.1) Use the GAP commands `Size` and `ulist` (see Chapters 1 and 2 of the lab manual) to determine the order of the group $U(p^a)$ for various odd primes p and positive integers a. For example, taking $p = 3$ and $a = 1$ means you compute the order of $U(3)$, and taking $p = 5$ and $a = 3$ means you compute the order of $U(125)$. Do enough examples so that you feel comfortable making a conjecture about the order of $U(p^a)$ where p is an odd prime and a is a positive integer. Carefully state your conjecture and do a few more examples to make sure you believe it. (You do not need to prove your conjecture.) Then use GAP to decide whether your conjecture applies also to the group $U(2^a)$ where a is a positive integer. Explain. To get started, you will want to load `ulist` by typing in

```
gap> ulist:= function(n)
> local s,i,o;
> o:= One(Integers mod n);
> s:= n-> Filtered([1..n-1], i -> Gcd(i,n) = 1);
> return s(n)*o;
> end;
```

(2) (Lab Manual, Chapter 2 Exercises, #2.5) Read about the command `GL(n,p)` in Chapter 2 of the lab manual. Use this command to find the order of $GL(2, \mathbb{Z}_p)$ and $SL(2, \mathbb{Z}_p)$ for $p = 3, 5, 7$ and 11. What relationship do you see between the orders of $GL(2, \mathbb{Z}_p)$ and $SL(2, \mathbb{Z}_p)$ and $p - 1$? Does this relationship hold for $p = 2$? Based on these examples, does it appear that p always divides the order of $SL(2, \mathbb{Z}_p)$? What about $p - 1$? What about $p + 1$? Guess a formula for the order of $SL(2, \mathbb{Z}_p)$. Guess a formula for the order of $GL(2, \mathbb{Z}_p)$.