Problem Set 3

Due: At the Beginning of Class on Thursday, February 27

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets.*

(1) (Gallian, Chapter 5 Exercises, #2a) Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}.$$

Write α, β and $\alpha\beta$ as products of disjoint cycles.

(2) (Gallian, Chapter 5 Exercises, #5c and e) What is the order of each of the following permutations?

c. $(1\ 2\ 4)(3\ 5)$

e. $(1\ 2\ 3\ 5)(2\ 4\ 5\ 6\ 7)$

(3) (Gallian, Chapter 5 Exercises, #13) Suppose that α is a mapping from a set S to itself and $\alpha(\alpha(x)) = x$ for all x in S. Prove that α is one-to-one and onto.

(4) (Gallian, Chapter 5 Exercises, #22) If α and β are distinct 2-cycles, what are the possibilities for $|\alpha\beta|$?

(5) (Gallian, Chapter 5 Exercises, #34) What cycle is $(a_1 a_2 \cdots a_n)^{-1}$?

(6) (Gallian, Chapter 3 Exercises, #35) Let G be a group of permutations on a set X. Let $a \in X$ and define $\text{stab}(a) = \{\alpha \in G \mid \alpha(a) = a\}$. We call $\text{stab}(a)$ the *stabilizer of a in G* (since it consists of all members of G that leave a fixed). Prove that $\text{stab}(a)$ is a subgroup of G. (This subgroup was introduced by Galois in 1832.)

(7) Let $\beta = (3\ 5)(2\ 3\ 5\ 4)(1\ 2\ 3\ 4) \in S_7$. Write β^{2009} as a product of disjoint cycles. Be sure to justify your answer. You must do this problem without the help of a computer.

Extra Credit GAP Exercises: This next section is not required. However, you may choose to complete the tasks for extra credit on this Problem Set. In order to receive credit, you must use GAP.

Please intersperse your GAP commands and output with your explanations. As usual, you should create a log file. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions. The GAP lab manual can be found at (`http://math.slu.edu/~rainbolt/FullManual8th.pdf`.

(1) Use the GAP commands `Size`, `ulist`, and `cyclic` (see Chapters 2 & 3 of the lab manual) to determine whether the group $U(n)$ is cyclic for various values of n of the form p^a where p is an odd prime and a is a positive integer. Do enough examples so that you feel comfortable making a conjecture, carefully state your conjecture, and then do a few more examples to make sure you believe it. (You do not need to prove your conjecture.) Then use GAP to decide whether your conjecture applies also to the group $U(2^a)$ where a is a positive integer. Explain. Do not use the command `IsCyclic` for this exercise.

Note: To get started, you will want to load `ulist` and `cyclic`. To see exactly what to type to load these commands, see the Appendices at the end of Chapters 2 and 3 of the lab manual. Observe that

```
Size(cyclic(p^a,x))
```

outputs the order of the cyclic subgroup $\langle x \rangle$ of $U(p^a)$.

(2) (Lab Manual, Chapter 4 Exercises, #4.2) Let $G = \langle a \rangle$ be the cyclic group of order 30 generated by the element a. By the Fundamental Theorem of Cyclic Groups there is exactly one subgroup of G of order k for each k that divides 30. In addition, by the Fundamental Theorem of Cyclic Groups, every subgroup of a cyclic group is cyclic. So, this subgroup of order k must be cyclic. Use GAP to find a generator for the smallest subgroup H of G containing:

(a) a^4 and a^6
(b) a^{10} and a^2
(c) a^{15} and a^2
(d) a^9 and a^{12}
(e) a^8 and a^{12}

Fill in the blank in the following conjecture: If $G = \langle a \rangle$ is a cyclic group of order n, then the smallest subgroup containing the elements a^i and a^j is $\langle a^t \rangle$, where $t =$ ______________. You do not need to prove your conjecture. (Do more examples if you need to.)

Note: Start by setting up the group $G = c30$ as the cyclic group of order 30, and the element a as a generator of the group by typing:

```
gap> c30 := CyclicGroup(IsPermGroup,30);

gap> a := c30.1;
```

We are representing a as a 30-cycle. You can now find the order of the smallest subgroup of G containing a^4 and a^6 by typing

```
gap> Size(Subgroup(c30, [a^4,a^6]));
```