Problem Set 4
Due: At the Beginning of Class on Thursday, March 20

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets*.

(1) (Gallian, Chapter 6 Exercises, #3) Let $\mathbb{R}^+$ be the group of positive real numbers under multiplication. Show that the mapping $\phi(x) = \sqrt{x}$ is an automorphism of $\mathbb{R}^+$.

(2) (Gallian, Chapter 6 Exercises, #4) Show that $U(8)$ is not isomorphic to $U(10)$.

(3) (Gallian, Chapter 6 Exercises, #10) Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all g in G is an automorphism if and only if G is Abelian.

(4) (Gallian, Chapter 6 Exercises, #14) Find $\text{Aut}(\mathbb{Z}_6)$.

(5) (Gallian, Chapter 6 Exercises, #29) If ϕ and γ are isomorphisms from the cyclic group $\langle a \rangle$ to some group and $\phi(a) = \gamma(a)$, prove that $\phi = \gamma$.

(6) (Gallian, Chapter 6 Exercises, #32) Prove property 4 of Theorem 6.3: Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. If K is a subgroup of G, then prove that

$$\phi(K) = \{\phi(k) \mid k \in K\}$$

is a subgroup of $\overline{G}$.

(7) (Gallian, Chapter 6 Exercises, #34) Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

(8) (Gallian, Chapter 6 Exercises, #41) Consider the following statement: The order of a subgroup divides the order of a group. Suppose you could prove this for finite permutation groups. Would the statement then be true for all finite groups? Explain.

(9) (Gallian, Chapter 6 Exercises, #48) Let ϕ be an isomorphism from a group G to a group $\overline{G}$ and let a belong to G. Prove that $\phi(C(a)) = C(\phi(a))$.

(10) Let G be a group. Prove that $|\text{Inn}(G)| = 1$ if and only if G is Abelian.

Extra Credit GAP Exercises: This next section is not required. However, you may choose to complete the tasks for extra credit on this Problem Set. In order to receive credit, you must use GAP.

Please intersperse your GAP commands and output with your explanations. As usual, you should create a log file. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions. The GAP lab manual can be found at (`http://math.slu.edu/~rainbolt/FullManual8th.pdf`.

(1) (Lab Manual, Chapter 5 Exercises, #5.8 - 5.11) The cycle structure of a permutation is the number of 2-cycles, 3-cycles, etc. it contains when it is written as the product of disjoint cycles. For example, (1 2 3)(4 5) and (1 3 6)(2 7) have the same cycle structure.

   (a) Pick a random element a of S_9 by typing in

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   a:=Random(SymmetricGroup(9));
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   Compare the cycle structure of a to the cycle structure of the permutation bab^{-1} for

   - $b = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$
   - $b = (1\ 2)(3\ 4)(5\ 6)(7\ 8)$
   - $b = (1\ 2\ 3\ 4)(5\ 6\ 7\ 8)$.

   (b) Repeat part (a) for a different element a in S_9.

   (c) Given two elements a and b in a group of permutations G, make a conjecture of how the cycle structures of a and bab^{-1} are related. Test your conjecture for a pair of elements in a dihedral group D_n of your choice.

   (d) Based on your conjecture in part (c), make a conjecture about a relationship between the order of an element a and the order of bab^{-1}.