Problem Set 5

Due: At the Beginning of Class on Thursday, April 3

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem and Quiz Sets*.

(1) (Gallian, Chapter 7 Exercises, #45c) Let

$$G = \{(1), (1\ 2)(3\ 4), (1\ 2\ 3\ 4)(5\ 6), (1\ 3)(2\ 4), (1\ 4\ 3\ 2)(5\ 6), (5\ 6)(1\ 3), (1\ 4)(2\ 3), (2\ 4)(5\ 6)\}.$$

Find the stabilizer of 5 and the orbit of 5.

(2) (Gallian, Chapter 7 Exercises, #62c) Use the Orbit-Stabilizer Theorem to calculate the order of the group of rotations of a regular dodecahedron (a solid with 12 congruent pentagons as faces).

(3) Subgroups and External Direct Products:

(a) Let G_1 and G_2 be groups, let H_1 be a subgroup of G_1, and let H_2 be a subgroup of G_2. Prove that $H_1 \oplus H_2$ is a subgroup of $G_1 \oplus G_2$.

(b) Find a subgroup of $\mathbb{Z}_9 \oplus \mathbb{Z}_3$ that is not of the form $H \oplus K$ for any subgroup H of $\mathbb{Z}_9$ and any subgroup K of $\mathbb{Z}_3$.

(4) (Gallian, Chapter 8 Exercises, #4) Let G and H be groups. Show that $G \oplus H$ is Abelian if and only if G and H are Abelian. State the general case.

(5) (Gallian, Chapter 8 Exercises, #7) Let G_1 and G_2 be groups. Prove that $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$. State the general case.

(6) (Gallian, Chapter 8 Exercises, #17) Let G and H be groups. If $G \oplus H$ is cyclic, prove that G and H are cyclic. State the general case. (*Note: You may apply the results of Chapter 8, Exercise #3 from Gallian's text.*)

(7) (Gallian, Chapter 8 Exercises, #22) Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $\mathbb{Z}_{30} \oplus \mathbb{Z}_{20}$. Fully justify your answer.

(8) (Gallian, Chapter 8 Exercises, #52) Is $\mathbb{Z}_{10} \oplus \mathbb{Z}_{12} \oplus \mathbb{Z}_6 \approx \mathbb{Z}_{60} \oplus \mathbb{Z}_6 \oplus \mathbb{Z}_2$? Fully justify your answer.

(9) (Gallian, Chapter 8 Exercises, #55) How many isomorphisms are there from $\mathbb{Z}_{12}$ to $\mathbb{Z}_4 \oplus \mathbb{Z}_3$? Fully justify your answer.

Extra Credit GAP Exercises: This next section is not required. However, you may choose to complete the tasks for extra credit on this Problem Set. In order to receive credit, you must use GAP.

Please intersperse your GAP commands and output with your explanations. As usual, you should create a log file. If you type up your solutions, you can cut and paste from this log file into your solution file; please use a different font so it is easy to tell what is what. If you hand-write your solutions, you should still print out your log file; then physically cut and paste it into your solutions. The GAP lab manual can be found at (`http://math.slu.edu/~rainbolt/FullManual8th.pdf`.

(1) (Lab Manual, Chapter 7 Exercises, #7.1 - 7.4) Read about the commands Stabilizer(G, s) and Orbit(G, s) in the GAP lab manual. In this exercise you will work with these two commands and the Dihedral groups. Note that for a given k, to define $G = D_k$ in GAP you should type

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gap> G:=DihedralGroup(IsPermGroup,2k);
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at the command prompt.

(a) Find the number of elements in Orbit(G, s) for $G = D_{10}$ and $s = 1, 2, 3$, and 4.

(b) Find the number of elements in Stabilizer(G, s) for $G = D_{10}$ and $s = 1, 2, 3$, and 4.

(c) Repeat parts (a) and (b) for D_{49} and D_{50}.

(d) Make a conjecture about the number of elements in Stabilizer(D_k, s) and in Orbit(D_k, s).