Quiz Set 5
For Quiz on Thursday, April 10

Work all of the following problems. A subset of the problems and definitions from Chapter 9 (not including internal direct products) will be on Quiz 5 to be given April 10. Quizzes will be graded for correctness and clarity.

(1) (Gallian, Chapter 9 Exercises, #6) Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \mid a, b, d \in \mathbb{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbb{R})$? Fully justify your answer.

(2) (Gallian, Chapter 9 Exercises, #9) Prove that if H has index 2 in G, then H is normal in G.

(3) (Gallian, Chapter 9 Exercises, #12) Prove that a factor group of a cyclic group is cyclic.

(4) (Gallian, Chapter 9 Exercises, modified #13)

(a) Prove that a factor group of an Abelian group is Abelian.

(b) Let H be a normal subgroup of G. If H and G/H are Abelian, must G be Abelian? Either prove the statement or give a concrete example to show it is not true in general.

(5) (Gallian, Chapter 9 Exercises, #19) What is the order of the factor group $(\mathbb{Z}_{10} \oplus U(10))/\langle(2, 9)\rangle$? Fully justify your answer.

(6) (Gallian, Chapter 9 Exercises, #25) Let $G = U(32)$ and $H = \{1, 31\}$. The group G/H is isomorphic to one of $\mathbb{Z}_8, \mathbb{Z}_4 \oplus \mathbb{Z}_2$, or $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Determine which one by elimination. Fully justify your answer.

(7) (Gallian, Chapter 9 Exercises, #50) If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

(8) (Gallian, Chapter 9 Exercises, #54) Let $G = \{-1, 1, -i, i, -j, j, -k, k\}$, where $i^2 = j^2 = k^2 = -1, -i = (-1)i, 1^2 = (-1)^2 = 1, ij = -ji = k, jk = -kj = i$, and $ki = -ik = j$. See page 203 of the text for a visual description of these multiplication rules. The group G is called the group of *quaternions* and is used in many applications of group theory.

(a) Construct the Cayley table for G.

(b) Show that $H = \{1, -1\}$ is a normal subgroup of G.

(c) Construct the Cayley table for G/H. Is G/H isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2 \oplus \mathbb{Z}_2$?