Problem Set 5
Due: Wednesday, October 3

Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined in the *General Problem Set Guidelines Sheet*.

Unless otherwise stated, all problems can be found in the appropriate *Exercises* sections of the text (*Abstract Algebra* by D. Dummit and R. Foote, 3rd Edition).

- Section 8.2 # 1, 3, 6
- Let R be a P.I.D. and $D = R - \{0\}$. Prove that the field of fractions $D^{-1}R$ is also a P.I.D.
- For the following, let u be a universal side divisor in $\mathbb{Z}[i]$.
  - (a) Prove that any associate of u is also a universal side divisor in $\mathbb{Z}[i]$.
  - (b) Prove that there is an associate u' of u such that $u' = x + iy$ with $x \geq 0$ and $y > 0$.
  - (c) Prove that the complex conjugate $\overline{u}$ of u is also a universal side divisor in $\mathbb{Z}[i]$.
  - (d) Apply the definition of universal side divisor to $x = 1 + i$ to deduce that $N(u) = 2$ or $N(u) = 5$ (where N is the usual field norm defined in Section 7.1). Conclude that u is an associate of either $1 + i$, $2 + i$ or $1 + 2i$.
  - (e) Prove that $1 + i$ is a universal side divisor in $\mathbb{Z}[i]$.
  - (f) The Gaussian integer $2 + i$ is a universal side divisor in $\mathbb{Z}[i]$. Use this fact to explain why $1 + 2i$ is a universal side divisor in $\mathbb{Z}[i]$. (Hint: Show that $1 + 2i$ is an associate of $2 - i$.)
  - (g) List the universal side divisors in $\mathbb{Z}[i]$.
- In $\mathbb{Z}[\sqrt{5}]$, prove that $1 + \sqrt{5}$ is irreducible but not prime. Deduce that $\mathbb{Z}[\sqrt{5}]$ is not a unique factorization domain.