Lab 1
Preliminaries & Average Rates of Change

Preliminaries

Determine if each of the following statements are true or false. If false, find a specific counter-example.

(a) $(a+b)^n = a^n + b^n$ for all real numbers a and b and all positive integers $n \geq 2$.

(b) $\dfrac{1}{a+b} = \dfrac{1}{a} + \dfrac{1}{b}$ for all real numbers a and b such that $a, b, a+b \neq 0$.

(c) $\sqrt{a^2+b^2} = a+b$ for all real numbers a and b.

(d) $\sqrt{a^2} = a$ for all real numbers a.

(e) $\dfrac{a+b}{a} = 1 + \dfrac{b}{a}$ for all real numbers such that $a \neq 0$.

(f) $\dfrac{\sin(a)}{\sin(b)} = \dfrac{a}{b}$ for all real numbers such that $b \neq k\pi$ where k is an integer.

Rates of Change

For the following exercises, recall the following definition.

Definition: Let $y = f(x)$ be a function defined on the interval $I = [a, b]$. The *average rate of change of f on the interval I* is the fraction

$$f_{ARC}[a,b] = \frac{f(b) - f(a)}{b - a}.$$

(1) Consider the function $f(x) = x^2$.

(a) Complete the following table for $f_{ARC}[a, b]$.

$[a,b]$	$f_{ARC}[a,b]$	$[a,b]$	$f_{ARC}[a,b]$
$[1,2]$		$[0,1]$	
$[1,1.5]$		$[0.5,1]$	
$[1,1.1]$		$[0.9,1]$	
$[1,1.01]$		$[0.99,1]$	
$[1,1.001]$		$[0.999,1]$	

(b) Estimate the instantaneous rate of change of $y = f(x)$ with respect to x at $x = 1$.

(c) Estimate the slope of the tangent line to the curve $y = f(x)$ at the point $(1, f(1)) = (1, 1)$.

(2) Assume that a and h are real numbers and that $h > 0$. Show that

$$f_{ARC}[a, a+h] = \frac{f(a+h) - f(a)}{h}.$$

(3) Assume that a and h are real numbers and that $h > 0$. Verify each statement. In parts (a) - (f), what happens when h is very close to 0?

(a) If $f(x) = 4$, then
$$f_{ARC}[a, a+h] = 0.$$

(b) If $f(x) = x$, then
$$f_{ARC}[a, a+h] = 1.$$

(c) If $f(x) = x^2$, then
$$f_{ARC}[a, a+h] = 2a + h.$$

(d) If $f(x) = x^3$, then
$$f_{ARC}[a, a+h] = 3a^2 + 3ah + h^2.$$

(e) If $f(x) = \frac{1}{x}$, then
$$f_{ARC}[a, a+h] = -\frac{1}{a(a+h)}.$$

(f) If $f(x) = \sqrt{x}$, then
$$f_{ARC}[a, a+h] = \frac{1}{\sqrt{a+h} + \sqrt{a}}.$$

(g) For functions f and g,
$$(f+g)_{ARC}[a, a+h] = f_{ARC}[a, a+h] + g_{ARC}[a, a+h].$$

(h) For functions f and g,
$$(f \cdot g)_{ARC}[a, a+h] = (f(a+h) \cdot g_{ARC}[a, a+h]) + (g(a) \cdot f_{ARC}[a, a+h]).$$

(4) Suppose that the function f is increasing on $[a, b]$ (so that $u < v$ implies $f(u) < f(v)$ for all u, v in $[a, b]$).

(a) Prove that $f_{ARC}[a, c] > 0$ for all c in $(a, b]$.

(b) What can you say if f is decreasing on $[a, b]$?