Problem Set 1
Due: 3:00 p.m. on Wednesday, September 2

Instructions: Carefully read Sections 7.1 and 7.2 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity.

1. (Dummit and Foote #3) Let R be a ring with identity and let S be a subring of R containing the identity. Prove that if u is a unit in S then u is a unit in R. Show by example that the converse is false.

2. (Dummit and Foote #4) Prove that the intersection of any non-empty collection of subrings of a ring is also a subring.

3. (Dummit and Foote #7) The *center* of a ring R is $Z(R) = \{z \in R \mid zr = rz \text{ for all } r \in R\}$ (i.e., is the set of all elements that commute with every element of R).
   (a) Prove that the center of a ring is a subring that contains the identity.
   (b) Prove that the center of a division ring is a field.

4. (Dummit and Foote #11) Prove that if R is an integral domain and $x^2 = 1$ for some $x \in R$ then $x = \pm 1$.

5. (Dummit and Foote #14) An element x in a ring R is called *nilpotent* if $x^m = 0$ for some $m \in \mathbb{Z}^+$. Let x be a nilpotent element of the commutative ring R.
   (a) Prove that x is either zero or a zero divisor.
   (b) Prove that rx is nilpotent for all $r \in R$.
   (c) Prove that $1 + x$ is a unit in R.
   (d) Deduce that the sum of a nilpotent element and a unit is a unit.

6. (Dummit and Foote #15) A ring R is called a *Boolean ring* if $a^2 = a$ for all $a \in R$. Prove that every Boolean ring is commutative.

7. Let R be a ring and consider the polynomial ring $R[x]$.
   (a) Assume that R is commutative. Let $f \in R[x]$ be monic. Prove that f is a unit in $R[x]$ if and only if $f = 1_R$.
   (b) Give an example of a non-zero commutative ring R with identity having a non-constant polynomial $f \in R[x]$ that is a unit in $R[x]$. (In particular, the assumption that f is monic is necessary in part (a).)