Problem Set 10

Due: 3:00 p.m. on Wednesday, November 4

Instructions: Carefully read Sections 10.4 and 10.5 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, let R be a commutative ring with identity.

1. (Dummit and Foote §10.4 #20) Let $I = (2, x)$ be the ideal generated by 2 and x in the ring $R = \mathbb{Z}[x]$. Show that the element $2 \otimes 2 + x \otimes x$ in $I \otimes_R I$ is not a simple tensor, i.e., cannot be written as $a \otimes b$ for some $a, b \in I$.

2. (The Five Lemma) Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\ \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \downarrow{\scriptstyle \alpha_4} & & \downarrow{\scriptstyle \alpha_5} \\ B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5 \end{array}$$

   (a) Prove that if α_1 is surjective and α_2, α_4 are injective, then α_3 is injective.

   (b) Prove that if α_5 is injective and α_2, α_4 are surjective, then α_3 is surjective.

3. Consider the following commutative diagram of R-module homomorphisms with exact rows:

$$\begin{array}{ccccccccc} & & A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & 0 \\ & & \downarrow{\scriptstyle \alpha_1} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\ 0 & \longrightarrow & B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & & \end{array}$$

   (a) Prove that there is an exact sequence $\mathrm{Ker}(\alpha_1) \to \mathrm{Ker}(\alpha_2) \to \mathrm{Ker}(\alpha_3)$.

   (b) Prove that there is an exact sequence $B_1/\mathrm{Im}(\alpha_1) \to B_2/\mathrm{Im}(\alpha_2) \to B_3/\mathrm{Im}(\alpha_3)$.

4. Let L, M, and N be unitary R-modules. Let $f : M \to N$ be an R-module isomorphism. Prove that the map $f^* : \mathrm{Hom}_R(N, L) \to \mathrm{Hom}_R(M, L)$ is an isomorphism.