Problem Set 11
Due: 5:00 p.m. on Tuesday, November 10

Instructions: Carefully read Section 10.5 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, let R be a ring with identity.

1. (Dummit and Foote §10.5 #3) Let P_1 and P_2 be R-modules. Prove that $P_1 \oplus P_2$ is a projective R-module if and only if both P_1 and P_2 are projective. You may assume the fact that any direct sum of free R-modules is free.

2. (Dummit and Foote §10.5 #6) Prove that the following are equivalent:

   (i) Every R-module is projective.

   (ii) Every R-module is injective.

3. Assume that R is commutative and let M be an R-module. Prove that the following conditions are equivalent:

   (i) M is flat over R.

   (ii) For every injective R-module homomorphism $g' : N' \longrightarrow N$, the induced homomorphism $id_M \otimes g' : M \otimes_R N' \longrightarrow M \otimes_R N$ is injective.

   (iii) For every short exact sequence

   $$0 \longrightarrow N' \xrightarrow{g'} N \xrightarrow{g} N'' \longrightarrow 0$$

   of R-module homomorphisms, the induced sequence

   $$0 \longrightarrow M \otimes_R N' \stackrel{id_M \otimes g'}{\longrightarrow} M \otimes_R N \stackrel{id_M \otimes g}{\longrightarrow} M \otimes_R N'' \longrightarrow 0$$

   is exact.

4. Let R be a commutative ring and let $S \subseteq R$ be a multiplicatively closed subset. Let M be an R-module. You may assume the fact that $(S^{-1}R) \otimes_R M$ has a well-defined $(S^{-1}R)$-module structure given by

   $$\frac{r}{s}\left(\frac{r'}{s'} \otimes_R m\right) := \frac{rr'}{ss'} \otimes_R m.$$

   Prove that the R-module isomorphism $g : (S^{-1}R) \otimes_R M \to S^{-1}M$ given by

   $$\frac{r}{s} \otimes_R m \mapsto \frac{rm}{s}$$

   is a $(S^{-1}R)$-module isomorphism.