Problem Set 12

Due: 3:00 p.m. on Wednesday, November 18

Instructions: Carefully read Sections 11.1 and 11.2 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

1. (Dummit and Foote §11.1 #8) Let V be a vector space over the field F and let φ be a linear transformation of V to itself. A nonzero element $v \in V$ satisfying $\varphi(v) = \lambda v$ for some $\lambda \in F$ is called an *eigenvector* of φ with *eigenvalue* λ. Prove that for any fixed $\lambda \in F$ the collection of eigenvectors of φ with eigenvalue λ together with 0 forms a subspace of V.

2. (Dummit and Foote §11.1 #9) Let V be a vector space over the field F and let φ be a linear transformation of V to itself. Suppose for $i = 1, 2, \ldots, k$ that $v_i \in V$ is an eigenvector for φ with eigenvalue $\lambda_i \in F$ and that all the eigenvalues λ_i are distinct. Prove that $v_1, v_2, \ldots, v_k$ are linearly independent. Conclude that any linear transformation on an n-dimensional vector space has at most n distinct eigenvalues.

3. (Dummit and Foote §11.2 #38) Let A and B be square matrices. Prove that the trace of their Kronecker product is the product of their traces: $\mathrm{tr}(A \otimes B) = \mathrm{tr}(A)\mathrm{tr}(B)$. (Recall that the trace of a square matrix is the sum of its diagonal entries.)

4. (Dummit and Foote §11.2 #39) Let F be a subfield of K and let $\beta : V \to W$ be a linear transformation of finite dimensional vector spaces over F.

   (a) Prove that $1 \otimes \beta$ is a K-linear transformation from the vector space $K \otimes_F V$ to $K \otimes_F W$ over K. (Here 1 denotes the identity map from K to itself.)

   (b) Let $\mathcal{B} = \{v_1, \ldots, v_n\}$ and $\mathcal{E} = \{w_1, \ldots, w_m\}$ be bases of V and W respectively. Prove that the matrix of $1 \otimes \beta$ with respect to the bases $\{1 \otimes v_1, \ldots, 1 \otimes v_n\}$ and $\{1 \otimes w_1, \ldots, 1 \otimes w_m\}$ is the same as the matrix of β with respect to $\mathcal{B}$ and $\mathcal{E}$.