Problem Set 13

Due: 3:00 p.m. on Wednesday, November 25

Instructions: Carefully read Sections 11.3 and 11.4 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, let R be a non-zero commutative ring with identity and let A, B, C be matrices with entries in R such that A and B are square.

1. Assume that the ring R is a field. For this exercise, you may use the fact that a matrix in $M_{n\times n}(R)$ is invertible if and only if its determinant is non-zero. Prove that the following conditions are equivalent:
   (i) A is invertible;
   (ii) The columns of A are linearly independent;
   (iii) The rows of A are linearly independent.

2. Assume that A, B, C fit into a block matrix
$$T = \begin{bmatrix} A & 0 \\ C & B \end{bmatrix}.$$
   Prove that $\det(T) = \det(A)\det(B)$. (*Hint:* Expand along the top row and use induction.)

3. Let R be a field. Let $P \in M_{n\times n}(R)$ be an invertible matrix and set $A' = PAP^{-1}$. Let $\lambda \in R$. We define the *eigenspace* of A associated to λ to be
$$E_\lambda(A) = \{\text{eigenvectors of } A \text{ with eigenvalue } \lambda\} \cup \{0\}.$$
   Recall that on Problem Set 12 you showed that E_λ is a subspace of R^n.
   (i) Prove that v is an eigenvector of A with eigenvalue λ if and only if Pv is an eigenvector of A' with eigenvalue λ.
   (ii) Prove that λ is an eigenvalue of A if and only if λ is an eigenvalue of A'.
   (iii) Prove that for all $\lambda \in R$, there is an R-linear isomorphism $E_\lambda(A) \cong E_\lambda(A')$.

4. Let V be a finite dimensional vector space over the field F. Let S be any subset of V^* and define the annihilator of S to be $\operatorname{ann}(S) = \{v \in V \mid f(v) = 0 \text{ for all } f \in S\}$. You may assume that $\operatorname{ann}(S)$ is a subspace of V. Prove that if W^* is any subspace of V^*, then
$$\dim_F(W^*) + \dim_F(\operatorname{ann}(W^*)) = \dim_F(V).$$