Problem Set 3
Due: 3:00 p.m. on Wednesday, September 16

Instructions: Carefully read Sections 7.3 and 7.4 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity $1 \neq 0$.

1. (Dummit and Foote §7.3 #34) Let I and J be ideals of the ring R.
   (a) Prove that $I + J$ is the smallest ideal of R containing both I and J.
   (b) Prove that IJ is an ideal contained in $I \cap J$.
   (c) Give an example where $IJ \neq I \cap J$.
   (d) Prove that if R is commutative and if $I + J = R$ then $IJ = I \cap J$.
2. Let R be an integral domain.
   (a) (Dummit and Foote §7.4 #8) Prove that $(a) = (b)$ for some elements $a, b \in R$, if and only if $a = ub$ for some unit u of R.
   (b) Let $a, b, c \in R$ such that $a \neq 0$.
      (i) Prove that if $(ab) = (ac)$, then $(b) = (c)$.
      (ii) Prove that if $(a) = (ab)$, then b is a unit.
3. Let R be a commutative ring, and let I and J be ideals of R. Prove that $I \cup J$ is an ideal of R if and only if either $I \subseteq J$ or $J \subseteq I$.
4. Let R be a commutative ring and let A be the polynomial ring $A = R[x_1, \ldots, x_n]$. Consider the ideal $I = (x_1, \ldots, x_n) \subseteq A$. Prove that for each $d \in \mathbb{Z}^+$, the ideal I^d is generated by the set $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_1 + \cdots + a_n = d\}$.