Problem Set 6

Due: 3:00 p.m. on Wednesday, October 7

Instructions: Carefully read Sections 8.3, 9.1, and 9.2 of the textbook. Work all of the following problems. A subset of the problems will be graded. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity.

1. For the following, let u be a universal side divisor in $\mathbb{Z}[i]$.
   (a) Prove that any associate of u is also a universal side divisor in $\mathbb{Z}[i]$.
   (b) Prove that there is an associate u' of u such that $u' = x + iy$ with $x \geq 0$ and $y > 0$.
   (c) Prove that the complex conjugate $\overline{u}$ of u is also a universal side divisor in $\mathbb{Z}[i]$.
   (d) Apply the definition of universal side divisor to $x = 1 + i$ to deduce that $N(u) = 2$ or $N(u) = 5$ (where N is the usual field norm defined in Section 7.1). Conclude that u is an associate of either $1 + i$, $2 + i$ or $1 + 2i$.
   (e) Prove that $1 + i$ is a universal side divisor in $\mathbb{Z}[i]$.
   (f) The Gaussian integer $2 + i$ is a universal side divisor in $\mathbb{Z}[i]$. Use this fact to explain why $1 + 2i$ is a universal side divisor in $\mathbb{Z}[i]$. (Hint: Show that $1 + 2i$ is an associate of $2 - i$.)
   (g) List the universal side divisors in $\mathbb{Z}[i]$.
2. In $\mathbb{Z}[\sqrt{5}]$, prove that $1 + \sqrt{5}$ is irreducible but not prime. Deduce that $\mathbb{Z}[\sqrt{5}]$ is not a Unique Factorization Domain.
3. (Dummit and Foote §9.1 #13) Prove that the rings $F[x, y]/(y^2 - x)$ and $F[x, y]/(y^2 - x^2)$ are not isomorphic for any field F.
4. Let R be an integral domain and consider the polynomial ring $S = R[x_1, \ldots, x_n]$.
   (a) Let $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$ and $f \in S$. Use induction to prove that there are polynomials $q_1, \ldots, q_n$ such that $q_i = q_i(x_1, \ldots, x_i) \in R[x_1, \ldots, x_i]$ for $i = 1, \ldots, n$ and
   $$f = \left[\sum_i q_i \cdot (x_i - a_i)\right] + f(\mathbf{a}).$$
   (b) Let $Z \subseteq S$ and set $I := (Z)S$. Let $\mathbf{a} = (a_1, \ldots, a_n) \in R^n$, and consider the ideal $\mathcal{M} = (x_1 - a_1, \ldots, x_n - a_n)S$. Prove that the following conditions are equivalent:
      (i) $I \subseteq \mathcal{M}$.
      (ii) For all $f \in I$, we have $f(\mathbf{a}) = 0$.
      (iii) For all $f \in Z$, we have $f(\mathbf{a}) = 0$.

Bonus. (Dummit and Foote §9.2 #4) Let F be a finite field. Prove that $F[x]$ contains infinitely many primes.