Problem Set 1

Due: 12:00 p.m. on Friday, September 16

Instructions: Work all of the following problems. A subset of the problems will be graded. Your grade will also reflect your participation in solution presentations. Be sure to adhere to the expectations outlined on the sheet *Guidelines for Problem Sets*. Submit your solutions in-class or to Dr. Cooper's mailbox in the Department of Mathematics.

Exercises: For this Problem Set, assume that all rings are non-zero and contain an identity. We also let k be a field and $x_0, \ldots, x_n$ be indeterminates.

1. Recall that an ideal $I \subseteq R = k[x_0, \ldots, x_n]$ is said to be *homogeneous* if whenever f is in I then each homogeneous component of f is also in I. Prove that an ideal $I \subseteq R$ is homogeneous if and only if I can be generated by homogeneous polynomials. [*Hint:* For one direction, try induction on degree.]

2. Let h and d be positive integers such that $d \geq h$. Prove that $h^{<d>} = h$.

3. Fix $\mathcal{H} := (1, 4, 6, 9, 10, 13, 13, \ldots)$ and let $S := k[x_1, x_2, x_3, x_4]$. Does there exist a homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$? Provide two reasons for your answer: one using an O-sequence approach and one using an order ideal of monomials approach.

4. For this exercise we use the same notation that was set up in our discussion of lifting monomial ideals. Let $f = \mathbf{x}^{\boldsymbol{\alpha}} \in S = k[x_1, \ldots, x_n]$. Prove the following two facts:

   (a) $\overline{f}(\overline{\boldsymbol{\beta}}) = 0$ if and only if $\boldsymbol{\alpha} \not\leq \boldsymbol{\beta}$;

   (b) $\overline{f}(\overline{\boldsymbol{\gamma}}) = 0$ for all $\boldsymbol{\gamma}$ with $\deg(\boldsymbol{\gamma}) \leq \deg(\boldsymbol{\alpha})$ (except for $\boldsymbol{\alpha}$ itself).

5. Let $S = k[x_1, x_2]$, where k is an algebraically closed field of characteristic zero. Further, let $J \subseteq S$ be a homogeneous ideal such that $\sqrt{J} = (x_1, x_2)$. We define the *initial degree* of J, denoted $\alpha(J)$, to be the least degree of a non-zero homogeneous polynomial in J (i.e., $\alpha(J) := \min\{t \geq 0 \mid J_t \neq 0\}$).

   (a) Set $B = S/J$. Prove that

   $$H(B, t) = \begin{cases} t + 1 & \text{for } t < \alpha(J) \\ \leq \alpha(J) & \text{for } t \geq \alpha(J). \end{cases}$$

   (b) Let $V \subseteq S_t$ be a non-zero subspace of S_t. Denote by $S_1 V$ the subspace of S_{t+1} generated by $\{Lv \mid L \in S_1 \text{ and } v \in V\}$. Prove that

   $$\dim_k(S_1 V) \geq \dim_k V + 1.$$

   (c) Let $I \subseteq R = k[x_0, x_1, x_2]$ be a radical homogeneous ideal and, as above, define $\alpha := \alpha(I) = \min\{t \geq 0 \mid I_t \neq 0\}$. Assume that x_0 is not a zero-divisor on $A = R/I$. Note that $R/(I, x_0) \cong S/J$ where J the homogeneous ideal obtained by setting $x_0 = 0$ in the generators of I and suppose that $\sqrt{J} = (x_1, x_2)$. Prove that $\Delta H(A)$ has the form

   $$\Delta H(A) = \{1, 2, 3, \ldots, \alpha - 1, \alpha, \Delta H(A, \alpha), \Delta H(A, \alpha + 1), \ldots\}$$

   where $\alpha \geq \Delta H(A, \alpha) \geq \Delta H(A, \alpha + 1) \geq \Delta H(A, \alpha + 2) \geq \cdots$.