Problem Set 3

In-Class Presentations October 19 and 21

Instructions: Work all of the following problems. For this Problem Set, each student will present a solution to one exercise in-class. The presentations will be given on October 19 and 21. No written submissions will be collected. Your grade will reflect the correctness of your solution, your presentation, and answers to any questions that arise.

Exercises: For this Problem set, let $G = (V, E)$ be a graph with vertices $V = \{x_1, \ldots, x_n\}$ and set $S = k[x_1, \ldots, x_n]$.

1. Prove that a subset $W \subseteq V$ is an independent set if and only if $V \setminus W$ is a vertex cover. In particular, prove that W is a maximal independent set if and only if $V \setminus W$ is a minimal vertex cover.

2. Construct a graph H from G by appending an isolated vertex x to G. Let $I(G)$ and $I(H)$ denote the edge ideals of G and H, respectively. Prove that $(S/I(G))[x] = S[x]/I(H), \alpha_0(G) = \alpha_0(H)$ and $\beta_0(H) = \beta_0(G) + 1$.

3. If $e \in E$ we denote by $G \setminus \{e\}$ the spanning subgraph of G obtained by deleting e and keeping all the vertices of G.

   (a) Prove that if $\alpha_0(G \setminus \{e\}) < \alpha_0(G)$, then $\alpha_0(G) = \alpha_0(G \setminus \{e\}) + 1$.

   (b) Prove that if $\alpha_0(G \setminus \{e\}) < \alpha_0(G)$, then $\beta_0(G) = \beta_0(G \setminus \{e\}) - 1$.

4. (Reference: R. Villarreal) G is said to be *bipartite* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every edge in G has one vertex in V_1 and one vertex in V_2. Note that V_1 and V_2 are each independent sets. The pair (V_1, V_2) is called a *bipartition* of G. Let G be a bipartite graph without isolated vertices. Assume that G has a bipartition $V_1 = \{x_1, \ldots, x_r\}$ and $V_2 = \{y_1, \ldots, y_r\}$ such that:

   - $\{x_i, y_i\} \in E$ for all i;
   - if $\{x_i, y_j\} \in E$ and $\{x_j, y_k\} \in E$ and i, j, k are distinct, then $\{x_i, y_k\} \in E$.

   Prove that G is unmixed.

5. Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal generated in degree d. Prove that I has a linear resolution if and only if $\operatorname{reg}(I) = d$. [*Hint:* It might be helpful to show that $\beta_{i,j}(I) = 0$ for $j < i + d$.]